MATH 1553, JANKOWSKI
MIDTERM 3, FALL 2019

Name		**Section**	

Please **read all instructions** carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- As always, $e_1, \ldots, e_n$ are the standard unit coordinate vectors in $\mathbf{R}^n$.
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page. use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered the text.
- Good luck!

This page was intentionally left blank.

Problem 1. [10 points]

These problems are true or false. Circle **T** if the statement is *always* true. Otherwise, circle **F**. You do not need to justify your answer. Assume that all entries of matrices A and B are real numbers.

a) **T** **F** If A is a 3×3 matrix and its characteristic polynomial is $-\lambda^3 + 2\lambda^2 - 17\lambda$, then $\det(A) = 0$.

b) **T** **F** Suppose A is a 3×3 matrix with columns v_1, v_2, v_3. If $v_1 - v_2 + v_3 = 0$, then the determinant of A must be zero.

c) **T** **F** Suppose A and B are $n \times n$ matrices with the same characteristic polynomial. If A is diagonalizable, then B must also be diagonalizable.

d) **T** **F** Suppose A and B are $n \times n$ matrices. If $\det(A) = \det(B)$, then $\det(A - B) = 0$.

e) **T** **F** Suppose A is a 3×3 matrix and that v is an eigenvector of A corresponding to the eigenvalue $\lambda = -1$. Then $-2v$ must also be an eigenvector of A corresponding to the eigenvalue $\lambda = -1$.

Solution.

a) True. $\det(A) = \det(A - 0I) = -0^3 + 2 \cdot 0^2 - 17 \cdot 0 = 0$.

b) True. The columns of A are linearly dependent, so $\det(A) = 0$.

c) False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

d) False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ have determinant 1 but

$$\det(A - B) = \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4.$$

e) True. $A(-2v) = -2Av = -2(-v) = 2v$ so $A(-2v) = -(-2v)$. Alternatively, we could note that the (-1)-eigenspace is closed under scalar multiplication and $-2v \neq 0$, so $-2v$ is a (-1)-eigenvector.

Extra space for scratch work on problem 1

Problem 2.

[12 points]

Short answer. All parts are unrelated. You do not need to show your work. In each part, all entries of all matrices are real numbers.

a) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 2$. Fill in the blank:

$$\det\begin{pmatrix} a & b & c \\ g & h & i \\ -2d+3g & -2e+3h & -2f+3i \end{pmatrix} = \underline{\hspace{3cm}}.$$

b) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that reflects across the line $y = -2x$, and let A be the standard matrix for T. Which of the following are true? Circle all that apply.

(i) The 1-eigenspace of A is $\operatorname{Span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$.

(ii) A is diagonalizable.

(iii) $\det(A+I) = 0$.

c) Write a 2×2 matrix A that is diagonalizable but not invertible.

d) Suppose A is a 3×3 matrix and its characteristic polynomial is

$$\det(A - \lambda I) = -(\lambda - 5)(\lambda - 3)^2.$$

Which of the following *must* be true? Circle all that apply.

(i) The 5-eigenspace of A has dimension 1.

(ii) The homogeneous system given by the equation $(A - 3I)x = 0$ has two free variables.

(iii) For each b in $\mathbf{R}^3$, the equation $Ax = b$ is consistent.

e) Is there a 3×3 matrix A with the property that the 2-eigenspace of A is $\mathbf{R}^3$? If your answer is yes, write such a matrix A.

Solution.

a) This matrix is obtained from the original by switching the last two rows (multiplying the determinant by -1), scaling the new last row by -2, and then adding 3 times row two to row three (no change). Thus the determinant is $2 \cdot -1 \cdot -2 = 4$.

b) All of them are true! We see (i) is true since T fixes all vectors along the line $y = -2x$, and (ii) is true because A is a 2×2 matrix with two distinct eigenvalues 1 and -1. Also, (iii) is true because -1 is an eigenvalue of A (recall that A flips all vectors perpendicular to the line $y = -2x$).

c) Many examples possible, for example $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

d) (i) and (iii) are true. We see (i) is true because 5 has algebraic multiplicity 1 thus has geometric multiplicity 1. For (ii), if $(A - 3I)x = 0$ has two free variables then the *geometric* mutliplicity of $\lambda = 3$ is 2, which is not necessarily true. For (iii), A is invertible since 0 is not an eigenvalue of A, so $Ax = b$ is guaranteed to be consistent no matter what b is.

e) Yes. There is exactly one such matrix, namely $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Problem 3. [8 points]

Let $A = \begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix}$.

a) Find the characteristic polynomial of A and the complex eigenvalues of A. Simplify your eigenvalues completely.

Solution:

$$\det(A-\lambda I) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 6\lambda + 13.$$

The eigenvalues are

$$\lambda = \frac{6 \pm \sqrt{6^2 - 4(13)}}{2} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i.$$

b) For the eigenvalue of A with *negative* imaginary part, find a corresponding eigenvector v.

Solution: Let $\lambda = 3 - 2i$.

$$\left(A - \lambda I \mid 0\right) = \left(\begin{array}{cc|c} 2-(3-2i) & 5 & 0 \\ -1 & 4-(3-2i) & 0 \end{array}\right) = \left(\begin{array}{cc|c} -1+2i & 5 & 0 \\ -1 & 1+2i & 0 \end{array}\right).$$

By the 2×2 eigenvector trick, we know that since $A - \lambda I$ is not invertible, then and eigenvector is $\begin{pmatrix} -b \\ a \end{pmatrix}$ where $(a \ \ b)$ is the first row of $A - \lambda I$. Therefore, an eigenvector for λ is $v = \begin{pmatrix} -5 \\ -1+2i \end{pmatrix}$. Any nonzero complex scalar multiple of that is also an eigenvector, so another correct answer is

$$v = \begin{pmatrix} 5 \\ 1-2i \end{pmatrix}.$$

Alternatively, $(A - \lambda I \ \ 0) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & -1-2i & 0 \\ 0 & 0 & 0 \end{array}\right)$ which gives $v = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix}$.

c) Using your answer from (b), write an eigenvector w of A corresponding to the eigenvalue with *positive* imaginary part. You do not need to show your work on this part.

Solution: $w = \overline{v}$, so $w = \begin{pmatrix} -5 \\ -1-2i \end{pmatrix}$ (using first v above), or $w = \begin{pmatrix} 1-2i \\ 1 \end{pmatrix}$, etc.

Extra space for work on problem 3

Problem 4. [10 points]

Let $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix}$.

a) Find the eigenvalues of A.
Solution:

$$\begin{aligned}\det(A-\lambda I) &= (-1-\lambda)\Big[(3-\lambda)(-2-\lambda)+4\Big] \\ &= -(\lambda+1)[\lambda^2-\lambda-6+4] = -(\lambda+1)[\lambda^2-\lambda-2] \\ &= -(\lambda+1)(\lambda+1)(\lambda-2) = -(\lambda+1)^2(\lambda-2)\end{aligned}$$

The eigenvalues are $\lambda = -1$ and $\lambda = 2$.

b) For each eigenvalue of A, find a basis for the corresponding eigenspace.
Solution: <u>For $\lambda = -1$</u>:

$$\big(A+I \mid 0\big) = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus $x_1 = x_1$ (free), $x_2 = \frac{x_3}{2}$, and $x_3 = x_3$ (free).

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_3/2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix}. \quad \text{Basis} : \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} \right\}.$$

<u>For $\lambda = 2$:</u>

$$\big(A-2I \mid 0\big) = \left(\begin{array}{ccc|c} -3 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \quad \text{Basis} : \left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

c) Is A diagonalizable? If your answer is yes, write an invertible 3×3 matrix C and a diagonal matrix D so that $A = CDC^{-1}$. If your answer is no, justify why A is not diagonalizable.

Yes: $A = CDC^{-1}$ where $C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Extra space for work on problem 4

Problem 5.

[10 points]

Parts (a) and (b) are unrelated.

a) Suppose A and B are 4×4 matrices satisfying

$$\det(A) = 5, \quad \det(AB^{-1}) = 10.$$

Find $\det(-2B)$. Simplify your answer completely.

Solution:

$$\det(A)\det(B^{-1}) = 10, \quad \det(B^{-1}) = \frac{10}{\det(A)} = \frac{10}{5} = 2, \qquad \det(B) = \frac{1}{2}.$$

Therefore, $\det(-2B) = (-2)^4 \det(B) = 16 \cdot \dfrac{1}{2} = 8$.

b) Let $A = C \begin{pmatrix} -1 & 0 \\ 0 & 1/2 \end{pmatrix} C^{-1}$, where the columns of C are (in order) $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

(i) Write one nonzero vector v so that $A^n v$ approaches the zero vector as n gets very large (you do not need to show your work on part (i)).

Solution: This problem uses our geometric interpretation of diagonalization. Since $\{v_1, v_2\}$ is a basis for $\mathbf{R}^2$, we can write $v = x_1 v_1 + x_2 v_2$ for some scalars x_1 and x_2. Since $Av_1 = -v_1$ and $Av_2 = \frac{1}{2}v_2$, we get

$$Av = A(x_1v_1 + x_2v_2) = x_1Av_1 + x_2Av_2 = -x_1v_1 + \frac{1}{2}x_2v_2,$$

$$A^2v = A^2(x_1v_1 + x_2v_2) = A(-x_1v_1 + \frac{1}{2}x_2v_2) = (-1)^2x_1v_1 + \left(\frac{1}{2}\right)^2 x_2v_2,$$

and in general

$$A^n v = A^n(x_1v_1 + x_2v_2) = (-1)^n x_1v_1 + \left(\frac{1}{2}\right)^n x_2v_2.$$

As n gets huge, $(-1)^n x_1 v_1$ just keeps flipping between $-x_1v_1$ and x_1v_1, whereas $\left(\frac{1}{2}\right)^n x_2v_2$ approaches the origin.

Thus, as n gets large, $A^n v$ will approach the origin precisely when $v = 0 + x_2v_2$ for some scalar x_2. We were asked for a nonzero v, so $\boxed{v = \begin{pmatrix} 1 \\ -3 \end{pmatrix}}$ or any nonzero scalar multiple of $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

The nonzero scalar multiples of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are the *opposite* of what we want, since cv_1 (where c is a nonzero scalar) just keeps flipping between $-cv_1$ and cv_1, never getting closer to the origin as n gets large.

(ii) Find $A^{10}\begin{pmatrix}1\\2\end{pmatrix}$. Show your work!

Solution: We saw in (i) that $A^n\begin{pmatrix}1\\2\end{pmatrix} = (-1)^n\begin{pmatrix}1\\2\end{pmatrix}$, so

$$A^{10}\begin{pmatrix}1\\2\end{pmatrix} = (-1)^{10}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}.$$

Many students tried to multiply the entire thing out by finding C^{-1} and doing $CD^{10}C^{-1}\begin{pmatrix}1\\2\end{pmatrix}$, but this was unnecessary and almost always led to an algebraic error or an unsimplified final answer.

(iii) Clearly circle your answer: Is A^4 diagonalizable? $\boxed{\text{YES}}$ NO
(no justification required for (iii))

Solution: Yes. $A^{10} = CD^{10}C^{-1}$ where D^{10} is the diagonal matrix

$$D^{10} = \begin{pmatrix}1 & 0\\ 0 & (\frac{1}{2})^{10}\end{pmatrix}.$$

Extra space for work on problem 5