Chapter 2

Systems of Linear Equations: Geometry
We want to think about the *algebra* in linear algebra (systems of equations and their solution sets) in terms of *geometry* (points, lines, planes, etc).

\[
\begin{align*}
x - 3y &= -3 \\
2x + y &= 8
\end{align*}
\]

This will give us better insight into the properties of systems of equations and their solution sets.

**Remember:** I expect you to be able to draw pictures!
Section 2.1

Vectors
We have been drawing elements of $\mathbb{R}^n$ as points in the line, plane, space, etc. We can also draw them as arrows.

**Definition**

A **point** is an element of $\mathbb{R}^n$, drawn as a point (a dot).

A **vector** is an element of $\mathbb{R}^n$, drawn as an arrow. When we think of an element of $\mathbb{R}^n$ as a vector, we’ll usually write it vertically, like a matrix with one column:

$$v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

The difference is purely psychological: *points and vectors are just lists of numbers.*
So why make the distinction?

A vector need not start at the origin: *it can be located anywhere!* In other words, an arrow is determined by its length and its direction, not by its location.

These arrows all represent the vector \( (1, 2) \).

However, unless otherwise specified, we’ll assume a vector starts at the origin.
Vector Algebra

Definition

- We can add two vectors together:

\[
\begin{pmatrix}
  a \\
  b \\
  c
\end{pmatrix} +
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} =
\begin{pmatrix}
  a + x \\
  b + y \\
  c + z
\end{pmatrix}.
\]

- We can multiply, or scalar, a vector by a real number \( c \):

\[
c
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} =
\begin{pmatrix}
  c \cdot x \\
  c \cdot y \\
  c \cdot z
\end{pmatrix}.
\]

We call \( c \) a scalar to distinguish it from a vector. If \( v \) is a vector and \( c \) is a scalar, \( cv \) is called a scalar multiple of \( v \).

(And likewise for vectors of length \( n \).) For instance,

\[
\begin{pmatrix}
  1 \\
  2 \\
  3
\end{pmatrix} +
\begin{pmatrix}
  4 \\
  5 \\
  6
\end{pmatrix} =
\begin{pmatrix}
  5 \\
  7 \\
  9
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
  -2
\end{pmatrix} 
\begin{pmatrix}
  1 \\
  2 \\
  3
\end{pmatrix} =
\begin{pmatrix}
  -2 \\
  -4 \\
  -6
\end{pmatrix}.
\]
The parallelogram law for vector addition

Geometrically, the sum of two vectors \( \mathbf{v}, \mathbf{w} \) is obtained as follows: place the tail of \( \mathbf{w} \) at the head of \( \mathbf{v} \). Then \( \mathbf{v} + \mathbf{w} \) is the vector whose tail is the tail of \( \mathbf{v} \) and whose head is the head of \( \mathbf{w} \). Doing this both ways creates a parallelogram. For example,

\[
\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}.
\]

Why? The width of \( \mathbf{v} + \mathbf{w} \) is the sum of the widths, and likewise with the heights. [interactive]

Vector subtraction

Geometrically, the difference of two vectors \( \mathbf{v}, \mathbf{w} \) is obtained as follows: place the tail of \( \mathbf{v} \) and \( \mathbf{w} \) at the same point. Then \( \mathbf{v} - \mathbf{w} \) is the vector from the head of \( \mathbf{w} \) to the head of \( \mathbf{v} \). For example,

\[
\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.
\]

Why? If you add \( \mathbf{v} - \mathbf{w} \) to \( \mathbf{w} \), you get \( \mathbf{v} \). [interactive]

This works in higher dimensions too!
Scalar Multiplication: Geometry

Scalar multiples of a vector
These have the same *direction* but a different *length*.

Some multiples of $\mathbf{v}$.

\[ \begin{align*}
2\mathbf{v} &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
\mathbf{v} &= \begin{pmatrix} 2 \\ 4 \end{pmatrix} \\
-\frac{1}{2}\mathbf{v} &= \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix} \\
0\mathbf{v} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{align*} \]

All multiples of $\mathbf{v}$.

So the scalar multiples of $\mathbf{v}$ form a *line*. 

[interactive]
Linear Combinations

We can add and scalar multiply in the same equation:

\[ w = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p \]

where \( c_1, c_2, \ldots, c_p \) are scalars, \( v_1, v_2, \ldots, v_p \) are vectors in \( \mathbb{R}^n \), and \( w \) is a vector in \( \mathbb{R}^n \).

**Definition**
We call \( w \) a **linear combination** of the vectors \( v_1, v_2, \ldots, v_p \). The scalars \( c_1, c_2, \ldots, c_p \) are called the **weights** or **coefficients**.

**Example**

Let \( v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \) and \( w = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

What are some linear combinations of \( v \) and \( w \)?

- \( v + w \)
- \( v - w \)
- \( 2v + 0w \)
- \( 2w \)
- \( -v \)

[interactive: 2 vectors]  [interactive: 3 vectors]
Is there any vector in $\mathbb{R}^2$ that is not a linear combination of $v$ and $w$?

No: in fact, every vector in $\mathbb{R}^2$ is a combination of $v$ and $w$.

(The purple lines are to help measure how much of $v$ and $w$ you need to get to a given point.)
What are some linear combinations of $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$?

- $\frac{3}{2}v$
- $-\frac{1}{2}v$
- $\ldots$

What are all linear combinations of $v$?

All vectors $cv$ for $c$ a real number. I.e., all scalar multiples of $v$. These form a line.

**Question**

What are all linear combinations of $v = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$?

**Answer:** The line which contains both vectors.

What’s different about this example and the one on the poll? [interactive]
Section 2.2

Vector Equations and Spans
Solve the following system of linear equations:

\[
\begin{align*}
    x - y &= 8 \\
    2x - 2y &= 16 \\
    6x - y &= 3.
\end{align*}
\]

We can write all three equations at once as vectors:

\[
\begin{pmatrix}
    x - y \\
    2x - 2y \\
    6x - y
\end{pmatrix}
= 
\begin{pmatrix}
    8 \\
    16 \\
    3
\end{pmatrix}.
\]

We can write this as a linear combination:

\[
x \begin{pmatrix}
    1 \\
    2 \\
    6
\end{pmatrix}
+ y \begin{pmatrix}
    -1 \\
    -2 \\
    -1
\end{pmatrix}
= 
\begin{pmatrix}
    8 \\
    16 \\
    3
\end{pmatrix}.
\]

So we are asking:

**Question:** Is \( \begin{pmatrix}
    8 \\
    16 \\
    3
\end{pmatrix} \) a linear combination of \( \begin{pmatrix}
    1 \\
    2 \\
    6
\end{pmatrix} \) and \( \begin{pmatrix}
    -1 \\
    -2 \\
    -1
\end{pmatrix} \)?
Systems of Linear Equations
Continued

\[ x - y = 8 \]
\[ 2x - 2y = 16 \]
\[ 6x - y = 3 \]

matrix form
\[
\begin{pmatrix}
1 & -1 & 8 \\
2 & -2 & 16 \\
6 & -1 & 3 \\
\end{pmatrix}
\]

row reduce
\[
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & -9 \\
0 & 0 & 0 \\
\end{pmatrix}
\]
solution
\[
\begin{pmatrix}
x \\
y \\
\end{pmatrix}
= \begin{pmatrix}
-1 \\
-9 \\
\end{pmatrix}
\]

Conclusion:
\[
- \begin{pmatrix}
1 \\
2 \\
6 \\
\end{pmatrix} - 9 \begin{pmatrix}
-1 \\
-2 \\
-1 \\
\end{pmatrix} = \begin{pmatrix}
8 \\
16 \\
3 \\
\end{pmatrix}
\]

[interactive] ←− (this is the picture of a consistent linear system)

What is the relationship between the vectors in the linear combination and the matrix form of the linear equation? They have the same columns!

Shortcut: You can go directly between augmented matrices and vector equations.
Vector Equations and Linear Equations

Summary

The vector equation

\[ x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b, \]

where \( v_1, v_2, \ldots, v_p, b \) are vectors in \( \mathbb{R}^n \) and \( x_1, x_2, \ldots, x_p \) are scalars, has the same solution set as the linear system with augmented matrix

\[
\begin{pmatrix}
| & | & | \\
v_1 & v_2 & \cdots & v_p & b \\
| & | & |
\end{pmatrix},
\]

where the \( v_i \)'s and \( b \) are the columns of the matrix.

So we now have (at least) two equivalent ways of thinking about linear systems of equations:

1. Augmented matrices.
2. Linear combinations of vectors (vector equations).

The last one is more geometric in nature.
It is important to know what are all linear combinations of a set of vectors $v_1, v_2, \ldots, v_p$ in $\mathbb{R}^n$: it’s exactly the collection of all $b$ in $\mathbb{R}^n$ such that the vector equation (in the unknowns $x_1, x_2, \ldots, x_p$)

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$

has a solution (i.e., is consistent).

**Definition**

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbb{R}^n$. The span of $v_1, v_2, \ldots, v_p$ is the collection of all linear combinations of $v_1, v_2, \ldots, v_p$, and is denoted $\text{Span}\{v_1, v_2, \ldots, v_p\}$. In symbols:

$$\text{Span}\{v_1, v_2, \ldots, v_p\} = \{x_1 v_1 + x_2 v_2 + \cdots + x_p v_p \mid x_1, x_2, \ldots, x_p \text{ in } \mathbb{R}\}.$$ 

**Synonyms:** $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is the subset spanned by or generated by $v_1, v_2, \ldots, v_p$.

This is the first of several definitions in this class that you simply **must learn**. I will give you other ways to think about Span, and ways to draw pictures, but *this is the definition*. Having a vague idea what Span means will not help you solve any exam problems!
Now we have several equivalent ways of making the same statement:

1. A vector $b$ is in the span of $v_1, v_2, \ldots, v_p$.
2. The vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = b$$

has a solution.
3. The linear system with augmented matrix

$$\begin{pmatrix}
| & | & | & | \\
v_1 & v_2 & \cdots & v_p & b \\
| & | & | & |
\end{pmatrix}$$

is consistent.

Note: equivalent means that, for any given list of vectors $v_1, v_2, \ldots, v_p, b$, 

either all three statements are true, or all three statements are false.
Pictures of Span

Drawing a picture of $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is the same as drawing a picture of all linear combinations of $v_1, v_2, \ldots, v_p$.

[interactive: span of two vectors in $\mathbb{R}^2$]
Pictures of Span

in $\mathbb{R}^3$

Span{$v$}

Span{$v, w$}

Span{$u, v, w$}

[interactive: span of two vectors in $\mathbb{R}^3$]

[interactive: span of three vectors in $\mathbb{R}^3$]
How many vectors are in Span \( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \)?

A. Zero  
B. One  
C. Infinity

In general, it appears that \( \text{Span}\{v_1, v_2, \ldots, v_p\} \) is the smallest “linear space” (line, plane, etc.) containing the origin and all of the vectors \( v_1, v_2, \ldots, v_p \).

We will make this precise later.
The whole lecture was about drawing pictures of systems of linear equations.

- **Points** and **vectors** are two ways of drawing elements of $\mathbb{R}^n$. Vectors are drawn as arrows.
- Vector addition, subtraction, and scalar multiplication have geometric interpretations.
- A **linear combination** is a sum of scalar multiples of vectors. This is also a geometric construction, which leads to lots of pretty pictures.
- The **span** of a set of vectors is the set of all linear combinations of those vectors. It is also fun to draw.
- A system of linear equations is equivalent to a vector equation, where the unknowns are the coefficients of a linear combination.