Sections 2.7 and 2.9

Basis, Dimension, Rank and Basis Theorems
Recall: a subspace of $\mathbb{R}^n$ is the same thing as a span, except we haven’t computed a spanning set yet.

For example, $\text{Col } A$ and $\text{Nul } A$ for a matrix $A$.

There are lots of choices of spanning set for a given subspace.

Are some better than others?
What is the smallest number of vectors that are needed to span a subspace?

**Definition**
Let $V$ be a subspace of $\mathbb{R}^n$. A **basis** of $V$ is a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $V$ such that:

1. $V = \text{Span}\{v_1, v_2, \ldots, v_m\}$, and
2. $\{v_1, v_2, \ldots, v_m\}$ is linearly independent.

The number of vectors in a basis is the **dimension** of $V$, and is written $\text{dim } V$.

**Why** is a basis the smallest number of vectors needed to span?
Recall: *linearly independent* means that every time you add another vector, the span gets bigger.

Hence, if we remove any vector, the span gets *smaller*: so any smaller set can’t span $V$.

**Important**
A subspace has *many different* bases, but they all have the same number of vectors.
**Bases of \( \mathbb{R}^2 \)**

**Question**  
What is a basis for \( \mathbb{R}^2 \)?

We need two vectors that span \( \mathbb{R}^2 \) and are linearly independent. \( \{ e_1, e_2 \} \) is one basis.

1. They span: \( \begin{pmatrix} a \\ b \end{pmatrix} = ae_1 + be_2 \).
2. They are linearly independent because they are not collinear.

**Question**  
What is another basis for \( \mathbb{R}^2 \)?

Any two nonzero vectors that are not collinear. \( \{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \} \) is also a basis.

1. They span: \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) has a pivot in every row.
2. They are linearly independent: \( \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \) has a pivot in every column.
The unit coordinate vectors

\[ e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ldots, \quad e_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \]

are a basis for \( \mathbb{R}^n \).

1. They span: \( I_n \) has a pivot in every row.
2. They are linearly independent: \( I_n \) has a pivot in every column.

In general: \( \{v_1, v_2, \ldots, v_n\} \) is a basis for \( \mathbb{R}^n \) if and only if the matrix

\[
A = \begin{pmatrix} | & | & | \\ \vdots & v_1 & v_2 \ldots & v_n \\ | & | & | \end{pmatrix}
\]

has a pivot in every row and every column.

Sanity check: we have shown that \( \dim \mathbb{R}^n = n \).
Basis of a Subspace

Example

Let
\[ V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid x + 3y + z = 0 \right\} \quad \text{and} \quad B = \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}. \]

Verify that \( B \) is a basis for \( V \). (So \( \dim V = 2 \): it is a plane.)

0. In \( V \): both vectors are in \( V \) because
\[-3 + 3(1) + 0 = 0 \quad \text{and} \quad 0 + 3(1) + (-3) = 0.\]

1. Span: If \( \begin{pmatrix} x \\ y \\ z \end{pmatrix} \) is in \( V \), then \( y = -\frac{1}{3}(x + z) \), so
\[ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{-x}{3} \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} - \frac{z}{3} \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}. \]

2. Linearly independent:
\[ c_1 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = 0 \implies \begin{pmatrix} -3c_1 \\ c_1 + c_2 \\ -3c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies c_1 = c_2 = 0. \]
Basis for Nul $A$

**Fact**

The vectors in the parametric vector form of the general solution to $Ax = 0$ always form a basis for Nul $A$.

**Example**

\[
A = \begin{pmatrix}
1 & 2 & 0 & -1 \\
-2 & -3 & 4 & 5 \\
2 & 4 & 0 & -2
\end{pmatrix}
\]

\[
\text{rref } A = \begin{pmatrix}
1 & 0 & -8 & -7 \\
0 & 1 & 4 & 3 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

The vectors span Nul $A$ by construction (every solution to $Ax = 0$ has this form).

1. The vectors span Nul $A$ by construction (every solution to $Ax = 0$ has this form).
2. Can you see why they are linearly independent? (Look at the last two rows.)
Basis for Col A

Fact

The *pivot columns* of $A$ always form a basis for Col $A$.

**Warning:** I mean the pivot columns of the *original* matrix $A$, not the row-reduced form. (Row reduction changes the column space.)

**Example**

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & 3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix} \quad \xrightarrow{\text{rref}} \quad \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot columns = basis ⇔ pivot columns in rref

So a basis for Col $A$ is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}.$$  

**Why?** See slides on linear independence.
Basis Theorem

Let $V$ be a subspace of dimension $m$. Then:
- Any $m$ linearly independent vectors in $V$ form a basis for $V$.
- Any $m$ vectors that span $V$ form a basis for $V$.

**Upshot**

If you *already* know that $\dim V = m$, and you have $m$ vectors $\mathcal{B} = \{v_1, v_2, \ldots, v_m\}$ in $V$, then you only have to check *one* of

1. $\mathcal{B}$ is linearly independent, or
2. $\mathcal{B}$ spans $V$

in order for $\mathcal{B}$ to be a basis.

**Example:** any three linearly independent vectors form a basis for $\mathbb{R}^3$. 

The Rank Theorem

Recall:
- The **dimension** of a subspace \( V \) is the number of vectors in a basis for \( V \).
- A basis for the column space of a matrix \( A \) is given by the pivot columns.
- A basis for the null space of \( A \) is given by the vectors attached to the free variables in the parametric vector form.

**Definition**
The **rank** of a matrix \( A \), written \( \text{rank} \ A \), is the dimension of the column space \( \text{Col} \ A \). The **nullity** of \( A \), written \( \text{nullity} \ A \), is the dimension of the solution set of \( A x = 0 \).

**Observe:**
\[
\begin{align*}
\text{rank} \ A &= \dim \text{Col} \ A = \text{the number of columns with pivots} \\
\text{nullity} \ A &= \dim \text{Nul} \ A = \text{the number of free variables} \\
&= \text{the number of columns without pivots}.
\end{align*}
\]

**Rank Theorem**
If \( A \) is an \( m \times n \) matrix, then
\[
\text{rank} \ A + \text{nullity} \ A = n = \text{the number of columns of} \ A.
\]
In other words,
\[
\text{(dimension of column space)} + \text{(dimension of solution set)} = \text{(number of variables)}.
\]
The Rank Theorem

Example

\[
A = \begin{pmatrix}
1 & 2 & 0 & -1 \\
-2 & -3 & 4 & 5 \\
2 & 4 & 0 & -2
\end{pmatrix}
\xrightarrow{rref}
\begin{pmatrix}
1 & 0 & -8 & -7 \\
0 & 1 & 4 & 3 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

basis of \( \text{Col } A \)
free variables

A basis for \( \text{Col } A \) is

\[
\left\{ \begin{pmatrix} 1 \\ -2 \\ 2 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 3 \\ 4 \end{pmatrix} \right\},
\]

so \( \text{rank } A = \dim \text{Col } A = 2 \).

Since there are two free variables \( x_3, x_4 \), the parametric vector form for the solutions to \( Ax = 0 \) is

\[
x = x_3 \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix}
\xrightarrow{\text{basis for } \text{Nul } A}
\left\{ \begin{pmatrix} 8 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}.
\]

Thus \( \text{nullity } A = \dim \text{Nul } A = 2 \).

The Rank Theorem says \( 2 + 2 = 4 \).
True or False: If $A$ is a $10 \times 15$ matrix and there is a basis of $\text{Col } A$ consisting of 4 vectors, then there is a basis of $\text{Nul } A$ consisting of 6 vectors.

False: if $\text{rank } A = 4$ then $\text{nullity } A = 15 - 4 = 11$. 
Summary

- A **basis** of a subspace is a minimal set of spanning vectors.
- There are recipes for computing a basis for the column space and null space of a matrix.
- The **dimension** of a subspace is the number of vectors in any basis.
- The **basis theorem** says that if you already know that \( \dim V = m \), and you have \( m \) vectors in \( V \), then you only have to check if they span or they’re linearly independent to know they’re a basis.
- The **rank theorem** says the dimension of the column space of a matrix, plus the dimension of the null space, is the number of columns of the matrix.