Chapter 3

Linear Transformations and Matrix Algebra
Section 3.1

Matrix Transformations
Let $A$ be a matrix, and consider the matrix equation $b = Ax$. If we vary $x$, we can think of this as a \textit{function} of $x$.

Many functions in real life—the \textit{linear} transformations—come from matrices in this way.

It makes us happy when a function comes from a matrix, because then questions about the function translate into questions a matrix, which we can usually answer.

For this reason, we want to study matrices as functions.
Matrices as Functions

Change in Perspective. Let $A$ be a matrix with $m$ rows and $n$ columns. Let’s think about the matrix equation $b = Ax$ as a function.

- The independent variable (the input) is $x$, which is a vector in $\mathbb{R}^n$.
- The dependent variable (the output) is $b$, which is a vector in $\mathbb{R}^m$.

As you vary $x$, then $b = Ax$ also varies. The set of all possible output vectors $b$ is the column space of $A$. 

$\mathbb{R}^n \xrightarrow{b = Ax} \mathbb{R}^m$

[interactive 1] [interactive 2]
Matrices as Functions

Projection

\[ A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

In the equation \( Ax = b \), the input vector \( x \) is in \( \mathbb{R}^3 \) and the output vector \( b \) is in \( \mathbb{R}^3 \). Then

\[
A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.
\]

This is projection onto the \( xy \)-plane. Picture:
Matrices as Functions

Reflection

\[ A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]

In the equation \( Ax = b \), the input vector \( x \) is in \( \mathbb{R}^2 \) and the output vector \( b \) is in \( \mathbb{R}^2 \). Then

\[ A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}. \]

This is reflection over the y-axis. Picture:
Matrices as Functions

Dilation

\[ A = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} \]

In the equation \( Ax = b \), the input vector \( x \) is in \( \mathbb{R}^2 \) and the output vector \( b \) is in \( \mathbb{R}^2 \).

\[
A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.5x \\ 1.5y \end{pmatrix} = 1.5 \begin{pmatrix} x \\ y \end{pmatrix}.
\]

This is dilation (scaling) by a factor of 1.5. Picture:
Matrices as Functions

Identity

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

In the equation \( Ax = b \), the input vector \( x \) is in \( \mathbb{R}^2 \) and the output vector \( b \) is in \( \mathbb{R}^2 \).

\[
A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.
\]

This is the identity transformation which does nothing. Picture:
Matrices as Functions

Rotation

\[ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

In the equation \( Ax = b \), the input vector \( x \) is in \( \mathbb{R}^2 \) and the output vector \( b \) is in \( \mathbb{R}^2 \). Then

\[ A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}. \]

What is this? Let's plug in a few points and see what happens.

\[ A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \]
\[ A \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \]
\[ A \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \]

It looks like \textit{counterclockwise rotation by 90°}. 
Matrices as Functions

Rotation

\[
A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

In the equation \( Ax = b \), the input vector \( x \) is in \( \mathbb{R}^2 \) and the output vector \( b \) is in \( \mathbb{R}^2 \). Then

\[
A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.
\]
In §4.1 there are other examples of geometric transformations of $\mathbb{R}^2$ given by matrices. Please look them over.
We have been drawing pictures of what it looks like to multiply a matrix by a vector, as a function of the vector.

Now let’s go the other direction. Suppose we have a function, and we want to know, does it come from a matrix?

**Example**

For a vector \( x \) in \( \mathbb{R}^2 \), let \( T(x) \) be the counterclockwise rotation of \( x \) by an angle \( \theta \). Is \( T(x) = Ax \) for some matrix \( A \)?

If \( \theta = 90^\circ \), then we know \( T(x) = Ax \), where

\[
A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

But for general \( \theta \), it’s not clear.

Our next goal is to answer this kind of question.
Transformations

Vocabulary

Definition

A **transformation** (or **function** or **map**) from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is a rule \( T \) that assigns to each vector \( x \) in \( \mathbb{R}^n \) a vector \( T(x) \) in \( \mathbb{R}^m \).

- \( \mathbb{R}^n \) is called the **domain** of \( T \) (the inputs).
- \( \mathbb{R}^m \) is called the **codomain** of \( T \) (the outputs).
- For \( x \) in \( \mathbb{R}^n \), the vector \( T(x) \) in \( \mathbb{R}^m \) is the **image** of \( x \) under \( T \).
  
  Notation: \( x \mapsto T(x) \).

- The set of all images \( \{ T(x) \mid x \text{ in } \mathbb{R}^n \} \) is the **range** of \( T \).

Notation:

\[
T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \text{means} \quad T \text{ is a transformation from } \mathbb{R}^n \text{ to } \mathbb{R}^m.
\]

It may help to think of \( T \) as a “machine” that takes \( x \) as an input, and gives you \( T(x) \) as the output.
Many of the functions you know and love have domain and codomain \( \mathbb{R} \).

\[
\sin : \mathbb{R} \rightarrow \mathbb{R} \\
\sin(x) = \left( \frac{\text{the length of the opposite edge}}{\text{the hypotenuse of a right triangle with angle} x \text{ in radians}} \right)
\]

Note how I’ve written down the \textit{rule} that defines the function \( \sin \).

\[
f : \mathbb{R} \rightarrow \mathbb{R} \\
f(x) = x^2
\]

Note that “\( x^2 \)” is sloppy (but common) notation for a function: it doesn’t have a name!

You may be used to thinking of a function in terms of its graph.

The horizontal axis is the domain, and the vertical axis is the codomain.

This is fine when the domain and codomain are \( \mathbb{R} \), but it’s hard to do when they’re \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \)! You need five dimensions to draw that graph.
Suppose you are building a robot arm with three joints that can move its hand around a plane, as in the following picture.

Define a transformation \( f: \mathbb{R}^3 \to \mathbb{R}^2 \):

\[
f(\theta, \varphi, \psi) = \text{position of the hand at joint angles } \theta, \varphi, \psi.
\]

Output of \( f \): where is the hand on the plane.

This function does not come from a matrix; belongs to the field of inverse kinematics.
Matrix Transformations

Definition
Let $A$ be an $m \times n$ matrix. The **matrix transformation** associated to $A$ is the transformation

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \text{defined by} \quad T(x) = Ax.$$ 

In other words, $T$ takes the vector $x$ in $\mathbb{R}^n$ to the vector $Ax$ in $\mathbb{R}^m$.

For example, if $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $T(x) = Ax$ then

$$T \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -14 \\ -32 \end{pmatrix}.$$ 

- The **domain** of $T$ is $\mathbb{R}^n$, which is the number of columns of $A$.
- The **codomain** of $T$ is $\mathbb{R}^m$, which is the number of rows of $A$.
- The **range** of $T$ is the set of all images of $T$:

$$T(x) = Ax = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n.$$ 

This is the **column space** of $A$. It is a span of vectors in the codomain.
Matrix Transformations

Example

\[ A = \begin{pmatrix} -1 & 0 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} \quad T(x) = Ax \quad T : \mathbb{R}^2 \rightarrow \mathbb{R}^3. \]

Domain is: \( \mathbb{R}^2 \). Codomain is: \( \mathbb{R}^3 \). Range is: all vectors of the form

\[ T \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \]

which is \( \text{Col} \ A \).
The picture of a matrix transformation is the same as the pictures we’ve been drawing all along. Only the language is different. Let

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and let

$$T(x) = Ax,$$

so $$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$. Then

$$T \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix},$$

which is still is reflection over the y-axis. Picture:
Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $T(x) = Ax$, so $T: \mathbb{R}^2 \to \mathbb{R}^2$. ($T$ is called a shear.)

What does $T$ do to this sheep?

**Hint:** first draw a picture what it does to the box *around* the sheep.
We can think of $b = Ax$ as a transformation with input $x$ and output $b$.

There are vocabulary words associated to transformations: **domain**, **codomain**, **range**.

A transformation that comes from a matrix is called a **matrix transformation**.

In this case, the vocabulary words mean something concrete in terms of matrices.

We like transformations that come from matrices, because questions about those transformations turn into questions about matrices.