Chapter 5

Eigenvalues and Eigenvectors
Section 5.1

Eigenvalues and Eigenvectors
Motivation

In a population of rabbits:
1. half of the newborn rabbits survive their first year;
2. of those, half survive their second year;
3. their maximum life span is three years;
4. rabbits have 0, 6, 8 baby rabbits in their three years, respectively.

If you know the population one year, what is the population the next year?

\[ f_n = \text{first-year rabbits in year } n \]
\[ s_n = \text{second-year rabbits in year } n \]
\[ t_n = \text{third-year rabbits in year } n \]

The rules say:
\[
\begin{pmatrix}
0 & 6 & 8 \\
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
\end{pmatrix}
\begin{pmatrix}
f_n \\
s_n \\
t_n \\
\end{pmatrix}
= 
\begin{pmatrix}
f_{n+1} \\
s_{n+1} \\
t_{n+1} \\
\end{pmatrix}.
\]

Let \( A = \begin{pmatrix}
0 & 6 & 8 \\
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
\end{pmatrix} \) and \( v_n = \begin{pmatrix}
f_n \\
s_n \\
t_n \\
\end{pmatrix} \). Then \( A v_n = v_{n+1} \).
If you know $v_0$, what is $v_{10}$?

$$v_{10} = Av_9 = AAv_8 = \cdots = A^{10}v_0.$$  

This makes it easy to compute examples by computer:  

<table>
<thead>
<tr>
<th>$v_0$</th>
<th>$v_{10}$</th>
<th>$v_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>30189</td>
<td>61316</td>
</tr>
<tr>
<td>7</td>
<td>7761</td>
<td>15095</td>
</tr>
<tr>
<td>9</td>
<td>1844</td>
<td>3881</td>
</tr>
<tr>
<td>1</td>
<td>9459</td>
<td>19222</td>
</tr>
<tr>
<td>2</td>
<td>2434</td>
<td>4729</td>
</tr>
<tr>
<td>3</td>
<td>577</td>
<td>1217</td>
</tr>
<tr>
<td>4</td>
<td>28856</td>
<td>58550</td>
</tr>
<tr>
<td>7</td>
<td>7405</td>
<td>14428</td>
</tr>
<tr>
<td>8</td>
<td>1765</td>
<td>3703</td>
</tr>
</tbody>
</table>

What do you notice about these numbers?

1. Eventually, each segment of the population doubles every year: $Av_n = v_{n+1} = 2v_n$.

2. The ratios get close to $(16 : 4 : 1)$:

$$v_n = \text{(scalar)} \cdot \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}.$$  

Translation: 2 is an eigenvalue, and $\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$ is an eigenvector!
Eigenvectors and Eigenvalues

Definition
Let $A$ be an $n \times n$ matrix.

Eigenvalues and eigenvectors are only for square matrices.

1. An **eigenvector** of $A$ is a nonzero vector $v$ in $\mathbb{R}^n$ such that $Av = \lambda v$, for some $\lambda$ in $\mathbb{R}$. In other words, $Av$ is a multiple of $v$.

2. An **eigenvalue** of $A$ is a number $\lambda$ in $\mathbb{R}$ such that the equation $Av = \lambda v$ has a nontrivial solution.

If $Av = \lambda v$ for $v \neq 0$, we say $\lambda$ is the **eigenvalue for** $v$, and $v$ is an **eigenvector for** $\lambda$.

Note: Eigenvectors are by definition nonzero. Eigenvalues may be equal to zero.

This is the most important definition in the course.
Verifying Eigenvectors

Example

\[
A = \begin{pmatrix}
0 & 6 & 8 \\
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0
\end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix}
16 \\
4 \\
1
\end{pmatrix}
\]

Multiply:

\[
A\mathbf{v} = \begin{pmatrix}
0 & 6 & 8 \\
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0
\end{pmatrix} \begin{pmatrix}
16 \\
4 \\
1
\end{pmatrix} = \begin{pmatrix}
32 \\
8 \\
2
\end{pmatrix} = 2\mathbf{v}
\]

Hence \( \mathbf{v} \) is an eigenvector of \( A \), with eigenvalue \( \lambda = 2 \).

Example

\[
A = \begin{pmatrix}
2 & 2 \\
-4 & 8
\end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

Multiply:

\[
A\mathbf{v} = \begin{pmatrix}
2 & 2 \\
-4 & 8
\end{pmatrix} \begin{pmatrix}
1 \\
1
\end{pmatrix} = \begin{pmatrix}
4 \\
4
\end{pmatrix} = 4\mathbf{v}
\]

Hence \( \mathbf{v} \) is an eigenvector of \( A \), with eigenvalue \( \lambda = 4 \).
Poll

Which of the vectors

A. \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)  B. \( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \)  C. \( \begin{pmatrix} -1 \\ 1 \end{pmatrix} \)  D. \( \begin{pmatrix} 2 \\ 1 \end{pmatrix} \)  E. \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \)

are eigenvectors of the matrix \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \)?

What are the eigenvalues?

\[
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{eigenvector with eigenvalue 2}
\]
\[
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{eigenvector with eigenvalue 0}
\]
\[
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{eigenvector with eigenvalue 0}
\]
\[
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad \text{not an eigenvector}
\]
\[
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{is never an eigenvector}
\]
Verifying Eigenvalues

Question: Is \( \lambda = 3 \) an eigenvalue of \( A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} \)?

In other words, does \( Av = 3v \) have a nontrivial solution?

\[ \text{... does } Av - 3v = 0 \text{ have a nontrivial solution?} \]
\[ \text{... does } (A - 3I)v = 0 \text{ have a nontrivial solution?} \]

We know how to answer that! Row reduction!

\[
A - 3I = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}
\]

Row reduce:

\[
\begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}
\]

Parametric form: \( x = -4y \); parametric vector form: \(
\begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -4 \\ 1 \end{pmatrix}.
\)

Does there exist an eigenvector with eigenvalue \( \lambda = 3 \)? Yes! Any nonzero multiple of \( \begin{pmatrix} -4 \\ 1 \end{pmatrix} \). Check:

\[
\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} -4 \\ 1 \end{pmatrix}.
\]
**Definition**

Let $A$ be an $n \times n$ matrix and let $\lambda$ be an eigenvalue of $A$. The $\lambda$-**eigenspace** of $A$ is the set of all eigenvectors of $A$ with eigenvalue $\lambda$, plus the zero vector:

$$
\lambda\text{-eigenspace} = \{ \mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \lambda \mathbf{v} \}
$$

$$
= \{ \mathbf{v} \in \mathbb{R}^n \mid (A - \lambda I)\mathbf{v} = 0 \}
$$

$$
= \text{Nul}(A - \lambda I).
$$

Since the $\lambda$-eigenspace is a null space, it is a **subspace** of $\mathbb{R}^n$.

How do you find a basis for the $\lambda$-eigenspace? Parametric vector form!
Eigenspaces

Example

Find a basis for the 3-eigenspace of

$$A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}.$$

We have to solve the matrix equation $A - 3I_2 = 0$.

$$A - 3I_2 = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$$

$$\text{RREF} \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

parametric form $x = -4y$

parametric vector form $\begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

basis $\left\{ \begin{pmatrix} -4 \\ 1 \end{pmatrix} \right\}$. 
Find a basis for the 2-eigenspace of 

\[ A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}. \]

\[ A - 2I = \begin{pmatrix} 3/2 & 0 & 3 \\ -3 & 0 & -3 \\ -3 & 0 & -3 \end{pmatrix} \]

Row reduce \[ \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

Parametric form \[ x = -2z \]

Parametric vector form \[ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \]

Basis \[ \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}. \]
Eigenspaces

Example

Find a basis for the $\frac{1}{2}$-eigenspace of

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$ 

$$A - \frac{1}{2} I = \begin{pmatrix} 3 & 0 & 3 \\ -3/2 & 3 & -3 \\ 0 & -3/2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

row reduce

parametric form

$$\begin{cases} x = -z \\ y = z \end{cases}$$

parametric vector form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

basis

$$\begin{Bmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \end{Bmatrix}.$$
Eigenspaces
Example: picture

\[
A = \begin{pmatrix}
\frac{7}{2} & 0 & 3 \\
-\frac{3}{2} & 2 & -3 \\
-\frac{3}{2} & 0 & -1 \\
\end{pmatrix}.
\]

We computed bases for the \(2\)-eigenspace and the \(\frac{1}{2}\)-eigenspace:

\[
\text{2-eigenspace: } \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}
\]
\[
\text{1/2-eigenspace: } \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}
\]

Hence the \(2\)-eigenspace is a plane and the \(\frac{1}{2}\)-eigenspace is a line.
Let $A$ be an $n \times n$ matrix and let $\lambda$ be a number.

1. $\lambda$ is an eigenvalue of $A$ if and only if $(A - \lambda I)x = 0$ has a nontrivial solution, if and only if $\text{Nul}(A - \lambda I) \neq \{0\}$.

2. In this case, finding a basis for the $\lambda$-eigenspace of $A$ means finding a basis for $\text{Nul}(A - \lambda I)$ as usual, i.e. by finding the parametric vector form for the general solution to $(A - \lambda I)x = 0$.

3. The eigenvectors with eigenvalue $\lambda$ are the nonzero elements of $\text{Nul}(A - \lambda I)$, i.e. the nontrivial solutions to $(A - \lambda I)x = 0$. 
We’ve seen that finding eigenvectors for a given eigenvalue is a row reduction problem.

Finding all of the eigenvalues of a matrix is not a row reduction problem! We’ll see how to do it in general next time. For now:

Fact: The eigenvalues of a triangular matrix are the diagonal entries.

Why? \( \text{Nul}(A - \lambda I) \neq \{0\} \) if and only if \( A - \lambda I \) is not invertible, if and only if \( \det(A - \lambda I) = 0 \).

\[
\begin{pmatrix}
3 & 4 & 1 & 2 \\
0 & -1 & -2 & 7 \\
0 & 0 & 8 & 12 \\
0 & 0 & 0 & -3 \\
\end{pmatrix}
- \lambda I_4 =
\begin{pmatrix}
3 - \lambda & 4 & 1 & 2 \\
0 & -1 - \lambda & -2 & 7 \\
0 & 0 & 8 - \lambda & 12 \\
0 & 0 & 0 & -3 - \lambda \\
\end{pmatrix}
\]

The determinant is \((3 - \lambda)(-1 - \lambda)(8 - \lambda)(-3 - \lambda)\), which is zero exactly when \( \lambda = 3, -1, 8, \) or \(-3\).
A Matrix is Invertible if and only if Zero is not an Eigenvalue

Fact: $A$ is invertible if and only if 0 is not an eigenvalue of $A$.

Why?

0 is an eigenvalue of $A$ \iff $Ax = 0x$ has a nontrivial solution
\iff $Ax = 0$ has a nontrivial solution
\iff $A$ is not invertible.

invertible matrix theorem
Eigenvectors with Distinct Eigenvalues are Linearly Independent

Fact: If $v_1, v_2, \ldots, v_k$ are eigenvectors of $A$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Why? If $k = 2$, this says $v_2$ can’t lie on the line through $v_1$. But the line through $v_1$ is contained in the $\lambda_1$-eigenspace, and $v_2$ does not have eigenvalue $\lambda_1$.

In general: see §5.1 (or work it out for yourself; it’s not too hard).

Consequence: An $n \times n$ matrix has at most $n$ distinct eigenvalues.
We have a couple of new ways of saying “A is invertible” now:

**The Invertible Matrix Theorem**
Let $A$ be a square $n \times n$ matrix, and let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. $A$ is invertible.
2. $T$ is invertible.
3. The reduced row echelon form of $A$ is $I_n$.
4. $A$ has $n$ pivots.
5. $Ax = 0$ has no solutions other than the trivial one.
6. $	ext{Nul}(A) = \{0\}$.
7. nullity($A$) = 0.
8. The columns of $A$ are linearly independent.
9. The columns of $A$ form a basis for $\mathbb{R}^n$.
10. $T$ is one-to-one.
11. $Ax = b$ is consistent for all $b$ in $\mathbb{R}^n$.
12. $Ax = b$ has a unique solution for each $b$ in $\mathbb{R}^n$.
13. The columns of $A$ span $\mathbb{R}^n$.
14. $	ext{Col } A = \mathbb{R}^m$.
15. $	ext{dim Col } A = m$.
16. rank $A = m$.
17. $T$ is onto.
18. There exists a matrix $B$ such that $AB = I_n$.
19. There exists a matrix $B$ such that $BA = I_n$.
20. The determinant of $A$ is *not* equal to zero.
21. The number 0 is *not* an eigenvalue of $A$. 


Summary

- **Eigenvectors** and **eigenvalues** are the most important concepts in this course.

- Eigenvectors are by definition nonzero; eigenvalues may be zero.

- The eigenvalues of a triangular matrix are the diagonal entries.

- A matrix is invertible if and only if zero is not an eigenvalue.

- Eigenvectors with distinct eigenvalues are linearly independent.

- The $\lambda$-eigenspace is the set of all $\lambda$-eigenvectors, plus the zero vector.

- You can compute a basis for the $\lambda$-eigenspace by finding the parametric vector form of the solutions of $(A - \lambda I_n)x = 0$.