Supplemental problems: §§2.6, 2.7, 2.9, 3.1

1. Circle T if the statement is always true, and circle F otherwise. You do not need to explain your answer.
   a) If \{v_1, v_2, v_3, v_4\} is a basis for a subspace V of \( \mathbb{R}^n \), then \{v_1, v_2, v_3\} is a linearly independent set.
   b) The solution set of a consistent matrix equation \( Ax = b \) is a subspace.
   c) A translate of a span is a subspace.

Solution.

a) True. If \{v_1, v_2, v_3\} is linearly dependent then \{v_1, v_2, v_3, v_4\} is automatically linearly dependent, which is impossible since \{v_1, v_2, v_3, v_4\} is a basis for a subspace.

b) False. This is true if and only if \( b = 0 \), i.e., the equation is homogeneous, in which case the solution set is the null space of \( A \).

c) False. A subspace must contain 0.

2. True or false (justify your answer). Answer true if the statement is always true. Otherwise, answer false.
   a) There exists a 3 \times 5 matrix with rank 4.
   b) If \( A \) is an 9 \times 4 matrix with a pivot in each column, then \( \text{Nul} A = \{0\} \).
   c) There exists a 4 \times 7 matrix \( A \) such that nullity \( A = 5 \).
   d) If \( \{v_1, v_2, \ldots, v_n\} \) is a basis for \( \mathbb{R}^4 \), then \( n = 4 \).

Solution.

a) False. The rank is the dimension of the column space, which is a subspace of \( \mathbb{R}^3 \), hence has dimension at most 3.

b) True.

c) True. For instance,

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

d) True. Any basis of \( \mathbb{R}^4 \) has 4 vectors.
3. Find bases for the column space and the null space of

\[ A = \begin{pmatrix} 0 & 1 & -3 & 1 & 0 \\ 1 & -1 & 8 & -7 & 1 \\ -1 & -2 & 1 & 4 & -1 \end{pmatrix}. \]

Solution.

The RREF of \(( A | 0)\) is

\[
\begin{pmatrix}
1 & 0 & 5 & -6 & 1 & 0 \\
0 & 1 & -3 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

so \(x_3, x_4, x_5\) are free, and

\[
\begin{pmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5
\end{pmatrix} = \begin{pmatrix}
 -5x_3 + 6x_4 - x_5 \\
 3x_3 - x_4 \\
 x_3 \\
 x_4 \\
 x_5
\end{pmatrix} = x_3 \begin{pmatrix}
 -5 \\
 3 \\
 1 \\
 0 \\
 0
\end{pmatrix} + x_4 \begin{pmatrix}
 6 \\
 -1 \\
 0 \\
 1 \\
 0
\end{pmatrix} + x_5 \begin{pmatrix}
 -1 \\
 0 \\
 0 \\
 0 \\
 1
\end{pmatrix}.
\]

Therefore, a basis for \(\text{Nul} \ A\) is

\[
\begin{Bmatrix}
\begin{pmatrix}
 -5 \\
 3 \\
 1 \\
 0 \\
 0
\end{pmatrix}, \\
\begin{pmatrix}
 6 \\
 -1 \\
 0 \\
 1 \\
 0
\end{pmatrix}, \\
\begin{pmatrix}
 -1 \\
 0 \\
 0 \\
 0 \\
 1
\end{pmatrix}
\end{Bmatrix}.
\]

To find a basis for \(\text{Col} \ A\), we use the pivot columns as they were written in the original matrix \(A\), not its RREF. These are the first two columns:

\[
\begin{Bmatrix}
\begin{pmatrix}
 0 \\
 1 \\
 1 \\
 -2
\end{pmatrix}, \\
\begin{pmatrix}
 1 \\
 0 \\
 -1 \\
 -2
\end{pmatrix}
\end{Bmatrix}.
\]

4. Find a basis for the subspace \(V\) of \(\mathbb{R}^4\) given by

\[ V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid x + 2y - 3z + w = 0 \right\}. \]

Solution.

\(V\) is \(\text{Nul} \ A\) for the \(1 \times 4\) matrix \(A = (1 \ 2 \ -3 \ 1)\). The augmented matrix \(( A | 0) = (1 \ 2 \ -3 \ 1 | 0)\) gives \(x = -2y + 3z - w\) where \(y, z, w\) are free variables. The parametric vector form for the solution set to \(Ax = 0\) is

\[
\begin{pmatrix}
 x \\
 y \\
 z \\
 w
\end{pmatrix} = \begin{pmatrix}
 -2y + 3z - w \\
 y \\
 z \\
 w
\end{pmatrix} = y \begin{pmatrix}
 -2 \\
 1 \\
 0 \\
 0
\end{pmatrix} + z \begin{pmatrix}
 3 \\
 0 \\
 1 \\
 0
\end{pmatrix} + w \begin{pmatrix}
 -1 \\
 0 \\
 0 \\
 1
\end{pmatrix}.
\]
Therefore, a basis for $V$ is
\[
\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.
\]

5.  
   a) True or false: If $A$ is an $m \times n$ matrix and $\text{Nul}(A) = \mathbb{R}^n$, then $\text{Col}(A) = \{0\}$.

   b) Give an example of a $2 \times 2$ matrix whose column space is the same as its null space.

   c) True or false: For some $m$, we can find an $m \times 10$ matrix $A$ whose column span has dimension 4 and whose solution set for $Ax = 0$ has dimension 5.

Solution.

   a) If $\text{Nul}(A) = \mathbb{R}^n$ then $Ax = 0$ for all $x$ in $\mathbb{R}^n$, so the only element in $\text{Col}(A)$ is $\{0\}$. Alternatively, the rank theorem says
\[
\dim(\text{Col} \ A) + \dim(\text{Nul} \ A) = n \implies \dim(\text{Col} \ A) + n = n \implies \dim(\text{Col} \ A) = 0 \implies \text{Col} \ A = \{0\}.
\]

   b) Take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Its null space and column space are Span $\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}$.

   c) False. The rank theorem says that the dimensions of the column space (Col$A$) and homogeneous solution space (Nul$A$) add to 10, no matter what $m$ is.

6. Suppose $V$ is a 3-dimensional subspace of $\mathbb{R}^5$ containing \[
\begin{pmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \\ 1 \end{pmatrix} \]

   Is \[
\left\{ \begin{pmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ a basis for } V? \text{ Justify your answer.}
\]

Solution.

Yes. The Basis Theorem says that since we know $\dim(V) = 3$, our three vectors will form a basis for $V$ if and only if they are linearly independent.

Call the vectors $v_1, v_2, v_3$. It is very little work to show that the matrix $A = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ has a pivot in every column, so the vectors are linearly independent.

7.  
   a) Write a $2 \times 2$ matrix $A$ with rank 2, and draw pictures of Nul$A$ and Col$A$. 

b) Write a $2 \times 2$ matrix $B$ with rank 1, and draw pictures of Nul$B$ and Col$B$.

\[
A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\]
\[
\text{Nul } B = \quad \text{Col } B =
\]

(Add images of matrix grids and null and column spaces)

8. For each matrix $A$, describe what the transformation $T(x) = Ax$ does to $\mathbb{R}^3$ geometrically.

a)\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}

b)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}

c)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}

d)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}

Solution.

a) We compute

\[
T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ y \\ z \end{pmatrix}.
\]

This is the reflection over the $yz$-plane.

b) We compute

\[
T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}.
\]
This is projection onto the \( z \)-axis.

c) We compute
\[
T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ z \end{pmatrix}.
\]

This is the reflection over the \( xz \)-plane.

d) 
\[
T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x \\ 0 \end{pmatrix}.
\]

This is projection onto the \( xy \)-plane, followed by reflection over the line \( y = x \).