Supplemental problems: §§2.6, 2.7, 2.9, 3.1

1. Circle T if the statement is always true, and circle F otherwise. You do not need to explain your answer.
   a) If \( \{v_1, v_2, v_3, v_4\} \) is a basis for a subspace \( V \) of \( \mathbb{R}^n \), then \( \{v_1, v_2, v_3\} \) is a linearly independent set.
   b) The solution set of a consistent matrix equation \( Ax = b \) is a subspace.
   c) A translate of a span is a subspace.

2. True or false (justify your answer). Answer true if the statement is always true. Otherwise, answer false.
   a) There exists a \( 3 \times 5 \) matrix with rank 4.
   b) If \( A \) is an \( 9 \times 4 \) matrix with a pivot in each column, then \( \text{Nul}(A) = \{0\} \).
   c) There exists a \( 4 \times 7 \) matrix \( A \) such that nullity \( A = 5 \).
   d) If \( \{v_1, v_2, \ldots, v_n\} \) is a basis for \( \mathbb{R}^4 \), then \( n = 4 \).

3. Find bases for the column space and the null space of
   \[
   A = \begin{pmatrix}
   0 & 1 & -3 & 1 & 0 \\
   1 & -1 & 8 & -7 & 1 \\
   -1 & -2 & 1 & 4 & -1
   \end{pmatrix}.
   \]

4. Find a basis for the subspace \( V \) of \( \mathbb{R}^4 \) given by
   \[
   V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \mid x + 2y - 3z + w = 0 \right\}.
   \]

5. a) True or false: If \( A \) is an \( m \times n \) matrix and \( \text{Null}(A) = \mathbb{R}^n \), then \( \text{Col}(A) = \{0\} \).
   b) Give an example of a \( 2 \times 2 \) matrix whose column space is the same as its null space.
   c) True or false: For some \( m \), we can find an \( m \times 10 \) matrix \( A \) whose column span has dimension 4 and whose solution set for \( Ax = 0 \) has dimension 5.

6. Suppose \( V \) is a 3-dimensional subspace of \( \mathbb{R}^5 \) containing
   \[
   \begin{pmatrix}
   1 \\
   -4 \\
   0
   \end{pmatrix},
   \begin{pmatrix}
   1 \\
   0 \\
   -3
   \end{pmatrix}, \text{ and } \begin{pmatrix}
   9 \\
   8 \\
   1
   \end{pmatrix}.
   \]
Is \{ \begin{pmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \} a basis for \( V \)? Justify your answer.

7. a) Write a \( 2 \times 2 \) matrix \( A \) with rank 2, and draw pictures of \( \text{Nul} A \) and \( \text{Col} A \).

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

\[
\text{Nul} A = \begin{pmatrix} \ast & \ast \\ \ast & \ast \end{pmatrix}
\]

\[
\text{Col} A = \begin{pmatrix} \ast & \ast \end{pmatrix}
\]

b) Write a \( 2 \times 2 \) matrix \( B \) with rank 1, and draw pictures of \( \text{Nul} B \) and \( \text{Col} B \).

\[
B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

\[
\text{Nul} B = \begin{pmatrix} \ast & \ast \\ \ast & \ast \end{pmatrix}
\]

\[
\text{Col} B = \begin{pmatrix} \ast \end{pmatrix}
\]

c) Write a \( 2 \times 2 \) matrix \( C \) with rank 0, and draw pictures of \( \text{Nul} C \) and \( \text{Col} C \).

\[
C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

\[
\text{Nul} C = \begin{pmatrix} \ast & \ast \\ \ast & \ast \end{pmatrix}
\]

\[
\text{Col} C = \begin{pmatrix} \ast \end{pmatrix}
\]

(In the grids, the dot is the origin.)

8. For each matrix \( A \), describe what the transformation \( T(x) = Ax \) does to \( \mathbb{R}^3 \) geometrically.

a) \( \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)  

b) \( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)  

c) \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)  

d) \( \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)