Supplemental problems: Chapter 4, Determinants

1. If $A$ is an $n \times n$ matrix, is it necessarily true that $\det(-A) = -\det(A)$? Justify your answer.

Solution.
No. Since $\det(cA) = c^n \det(A)$, we see $\det(-A) = (-1)^n \det(A)$, so $\det(-A) = \det(A)$ if $n$ is even, and $\det(-A) = -\det(A)$ if $n$ is odd.

2. Let $A$ be an $n \times n$ matrix.
   a) Using cofactor expansion, explain why $\det(A) = 0$ if $A$ has a row or a column of zeros.
   b) Using cofactor expansion, explain why $\det(A) = 0$ if $A$ has adjacent identical columns.

Solution.
   a) If $A$ has zeros for all entries in row $i$ (so $a_{i1} = a_{i2} = \cdots = a_{in} = 0$), then the cofactor expansion along row $i$ is
      $$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = 0 \cdot C_{i1} + 0 \cdot C_{i2} + \cdots + 0 \cdot C_{in} = 0.$$  
      Similarly, if $A$ has zeros for all entries in column $j$, then the cofactor expansion along column $j$ is the sum of a bunch of zeros and is thus 0.

   b) If $A$ has identical adjacent columns, then the cofactor expansions will be identical except that one expansion’s terms for $\det(A)$ will have plus signs where the other expansion’s terms for $\det(A)$ have minus signs (due to the $(-1)^\text{power}$ factors) and vice versa.
      Therefore, $\det(A) = -\det(A)$, so $\det A = 0$.

3. Find the volume of the parallelepiped in $\mathbb{R}^4$ naturally determined by the vectors
   $$\begin{pmatrix} 4 \\ 1 \\ 3 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 7 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -5 \\ 0 \\ 7 \end{pmatrix}.$$

Solution.
We put the vectors as columns of a matrix $A$ and find $|\det(A)|$. For this, we expand $\det(A)$ along the third row because it only has one nonzero entry.

$$\det(A) = 3(-1)^{3+1} \cdot \det \begin{pmatrix} 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{pmatrix} = 3 \cdot 5(-1)^{1+3} \det \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix} = 3(5)(1)(7-6) = 15.$$  
(In the second step, we used the cofactor expansion along the first row since it had only one nonzero entry.)
The volume is $|\det(A)| = |15| = 15$.  

4. Let \( A = \begin{pmatrix} -1 & 1 \\ 1 & 7 \end{pmatrix} \), and define a transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( T(x) = Ax \). Find the area of \( T(S) \), if \( S \) is a triangle in \( \mathbb{R}^2 \) with area 2.

**Solution.**

\[ |\det(A)|\text{Vol}(S) = |-7 - 1| \cdot 2 = 16. \]

5. Let \( A = \begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & 1 & 5 & 4 \\ 1 & -1 & -3 & 0 \\ -1 & 0 & 5 & 4 \\ 3 & -3 & -2 & 5 \end{pmatrix} \)

a) Compute \( \det(A) \).

b) Compute \( \det(B) \).

c) Compute \( \det(AB) \).

d) Compute \( \det(A^2B^{-1}AB^2) \).

**Solution.**

a) Cofactor expansion would take some time, since the matrix has almost no zero entries. We use row reduction below, where \( r \) counts the row swaps and \( s \) measures the scaling factors.

\[
\begin{pmatrix}
2 & -8 & 6 & 8 \\
3 & -9 & 5 & 10 \\
-3 & 0 & 1 & -2 \\
1 & -4 & 0 & 6
\end{pmatrix}
\xrightarrow{R_1 = \frac{R_2}{2}}
\begin{pmatrix}
1 & -4 & 3 & 4 \\
3 & -9 & 5 & 10 \\
-3 & 0 & 1 & -2 \\
1 & -4 & 0 & 6
\end{pmatrix}
(r = 0, \ s = \frac{1}{2})
\]

\[
\xrightarrow{R_2 = R_2 - 3R_1}
\begin{pmatrix}
1 & -4 & 3 & 4 \\
0 & 3 & -4 & -2 \\
0 & -12 & 10 & 10 \\
0 & 0 & -3 & 2
\end{pmatrix}
(r = 0, \ s = \frac{1}{2})
\]

\[
\xrightarrow{R_3 = R_3 + 3R_1, \ R_4 = R_4 - R_3}
\begin{pmatrix}
1 & -4 & 3 & 4 \\
0 & 3 & -4 & -2 \\
0 & 0 & -6 & 2 \\
0 & 0 & -3 & 2
\end{pmatrix}
(r = 0, \ s = \frac{1}{2})
\]

\[
\xrightarrow{R_3 = R_3 + 4R_2}
\begin{pmatrix}
1 & -4 & 3 & 4 \\
0 & 3 & -4 & -2 \\
0 & 0 & -6 & 2 \\
0 & 0 & -3 & 2
\end{pmatrix}
(r = 0, \ s = \frac{1}{2})
\]

\[
\xrightarrow{R_4 = R_4 - R_2}
\begin{pmatrix}
1 & -4 & 3 & 4 \\
0 & 3 & -4 & -2 \\
0 & 0 & -6 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix}
(r = 0, \ s = \frac{1}{2})
\]

\[ \det(A) = (-1)^0 \frac{1 \cdot 3 \cdot (-6) \cdot 1}{1/2} = -36. \]
b) This is a complicated matrix without a lot of zeros, so we compute the determinant by row reduction. After one row swap and several row replacements, we reduce to the matrix
\[
\begin{pmatrix}
1 & -1 & -3 & 0 \\
0 & 1 & 5 & 4 \\
0 & 0 & 7 & 8 \\
0 & 0 & 0 & -3
\end{pmatrix}
\]. The determinant of this matrix is $-21$, so the determinant of the original matrix is 21.

c) $\det(AB) = \det(A)\det(B) = (-36)(21) = -756$.
d) $\det(A^2B^{-1}A^{-2}) = \det(A)^2\det(B)^{-1}\det(A)\det(B)^2 = \det(A)^3\det(B) = (-36)^3(21) = -979,776$.

6. If $A$ is a $3 \times 3$ matrix and $\det(A) = 1$, what is $\det(-2A)$?

**Solution.**
By determinant properties, scaling one row by $c$ multiplies the determinant by $c$. When we take $cA$ for an $n \times n$ matrix $A$, we are multiplying each row by $c$. This multiplies the determinant by $c$ a total of $n$ times.

Thus, if $A$ is $n \times n$, then $\det(cA) = c^n \det(A)$. Here $n = 3$, so
\[
\det(-2A) = (-2)^3\det(A) = -8 \det(A) = -8.
\]

7. a) Is there a real $2 \times 2$ matrix $A$ that satisfies $A^4 = -I_2$? Either write such an $A$, or show that no such $A$ exists.

(hint: think geometrically! The matrix $-I_2$ represents rotation by $\pi$ radians).

b) Is there a real $3 \times 3$ matrix $A$ that satisfies $A^4 = -I_3$? Either write such an $A$, or show that no such $A$ exists.

**Solution.**
a) Yes. Just take $A$ to be the matrix of counterclockwise rotation by $\frac{\pi}{4}$ radians:
\[
A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.
\]

Then $A^2$ gives rotation c.c. by $\frac{\pi}{2}$ radians, $A^3$ gives rotation c.c. by $\frac{3\pi}{4}$ radians, and $A^4$ gives rotation c.c. by $\pi$ radians, which has matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -I_2$.

b) No. If $A^4 = -I$ then
\[
[\det(A)]^4 = \det(A^4) = \det(-I) = (-1)^3 = -1.
\]
In other words, if $A^4 = -I$ then $[\det(A)]^4 = -1$, which is impossible since $\det(A)$ is a real number.

Similarly, $A^4 = -I$ is impossible if $A$ is $5 \times 5$, $7 \times 7$, etc.
1. True or false. Answer true if the statement is always true. Otherwise, answer false.
   a) If $A$ and $B$ are $n \times n$ matrices and $A$ is row equivalent to $B$, then $A$ and $B$ have the same eigenvalues.
   b) If $A$ is an $n \times n$ matrix and its eigenvectors form a basis for $\mathbb{R}^n$, then $A$ is invertible.
   c) If 0 is an eigenvalue of the $n \times n$ matrix $A$, then $\text{rank}(A) < n$.
   d) The diagonal entries of an $n \times n$ matrix $A$ are its eigenvalues.
   e) If $A$ is invertible and 2 is an eigenvalue of $A$, then $\frac{1}{2}$ is an eigenvalue of $A^{-1}$.
   f) If $\det(A) = 0$, then 0 is an eigenvalue of $A$.

Solution.

a) False. For instance, the matrices $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are row equivalent, but have different eigenvalues.

b) False. For example, the zero matrix is not invertible but its eigenvectors form a basis for $\mathbb{R}^n$.

c) True. If $\lambda = 0$ is an eigenvalue of $A$ then $A$ is not invertible so its associated transformation $T(x) = Ax$ is not onto, hence $\text{rank}(A) < n$.

d) False. This is true if $A$ is triangular, but not in general.

For example, if $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ then the diagonal entries are 2 and 0 but the only eigenvalue is $\lambda = 1$, since solving the characteristic equation gives us

$(2 - \lambda)(-\lambda) - (1)(-1) = 0 \quad \lambda^2 - 2\lambda + 1 = 0 \quad (\lambda - 1)^2 = 0 \quad \lambda = 1.

e) True. Let $v$ be an eigenvector corresponding to the eigenvalue 2.

$Av = 2v \implies A^{-1}Av = A^{-1}(2v) \implies v = 2A^{-1}v \implies \frac{1}{2}v = A^{-1}v.$

Therefore, $v$ is an eigenvector of $A^{-1}$ corresponding to the eigenvalue $\frac{1}{2}$.

f) True. If $\det(A) = 0$ then $A$ is not invertible, so $Av = 0v$ has a nontrivial solution.

2. In this problem, you need not explain your answers; just circle the correct one(s).

Let $A$ be an $n \times n$ matrix.

a) Which one of the following statements is correct?

1. An eigenvector of $A$ is a vector $v$ such that $Av = \lambda v$ for a nonzero scalar $\lambda$. 

2. An eigenvector of $A$ is a nonzero vector $v$ such that $Av = \lambda v$ for a scalar $\lambda$.

3. An eigenvector of $A$ is a nonzero scalar $\lambda$ such that $Av = \lambda v$ for some vector $v$.

4. An eigenvector of $A$ is a nonzero vector $v$ such that $Av = \lambda v$ for a nonzero scalar $\lambda$.

b) Which one of the following statements is not correct?

1. An eigenvalue of $A$ is a scalar $\lambda$ such that $A - \lambda I$ is not invertible.

2. An eigenvalue of $A$ is a scalar $\lambda$ such that $(A - \lambda I)v = 0$ has a solution.

3. An eigenvalue of $A$ is a scalar $\lambda$ such that $Av = \lambda v$ for a nonzero vector $v$.

4. An eigenvalue of $A$ is a scalar $\lambda$ such that $\det(A - \lambda I) = 0$.

Solution.

a) Statement 2 is correct: an eigenvector must be nonzero, but its eigenvalue may be zero.

b) Statement 2 is incorrect: the solution $v$ must be nontrivial.

3. Find a basis $B$ for the $(-1)$-eigenspace of $Z = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$

Solution.

For $\lambda = -1$, we find $\text{Nul}(Z - \lambda I)$.

\[
\begin{pmatrix} Z - \lambda I | 0 \end{pmatrix} = \begin{pmatrix} Z + I | 0 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Therefore, $x = -y$, $y = y$, and $z = 0$, so

\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.
\]

A basis is $B = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$. We can check to ensure $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector with corresponding eigenvalue $-1$:

\[
Z \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 + 3 \\ -3 + 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.
\]
4. Suppose $A$ is an $n \times n$ matrix satisfying $A^2 = 0$. Find all eigenvalues of $A$. Justify your answer.

**Solution.**

If $\lambda$ is an eigenvalue of $A$ and $v \neq 0$ is a corresponding eigenvector, then

$$Av = \lambda v \implies A(Av) = A\lambda v \implies A^2v = \lambda (Av) \implies 0 = \lambda (\lambda v) \implies 0 = \lambda^2 v.$$ 

Since $v \neq 0$ this means $\lambda^2 = 0$, so $\lambda = 0$. This shows that 0 is the only possible eigenvalue of $A$.

On the other hand, $\det(A) = 0$ since $(\det(A))^2 = \det(A^2) = \det(0) = 0$, so 0 must be an eigenvalue of $A$. Therefore, the only eigenvalue of $A$ is 0.

5. Match the statements (i)-(v) with the corresponding statements (a)-(e). All matrices are $3 \times 3$. There is a unique correspondence. Justify the correspondences in words.

(i) $Ax = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ has a unique solution.

(ii) The transformation $T(v) = Av$ fixes a nonzero vector.

(iii) $A$ is obtained from $B$ by subtracting the third row of $B$ from the first row of $B$.

(iv) The columns of $A$ and $B$ are the same; except that the first, second and third columns of $A$ are respectively the first, third, and second columns of $B$.

(v) The columns of $A$, when added, give the zero vector.

(a) 0 is an eigenvalue of $A$.
(b) $A$ is invertible.
(c) $\det(A) = \det(B)$
(d) $\det(A) = -\det(B)$
(e) 1 is an eigenvalue of $A$.

**Solution.**

(i) matches with (b).

(ii) matches with (e).

(iii) matches with (c).

(iv) matches with (d).

(v) matches with (a).