Supplemental problems: Chapter 4, Determinants

1. If $A$ is an $n \times n$ matrix, is it necessarily true that $\det(-A) = -\det(A)$? Justify your answer.

2. Let $A$ be an $n \times n$ matrix.
   a) Using cofactor expansion, explain why $\det(A) = 0$ if $A$ has a row or a column of zeros.
   b) Using cofactor expansion, explain why $\det(A) = 0$ if $A$ has adjacent identical columns.

3. Find the volume of the parallelepiped in $\mathbb{R}^4$ naturally determined by the vectors
   \[
   \begin{pmatrix}
   4 \\
   1 \\
   3 \\
   8 \\
   \end{pmatrix}, \quad
   \begin{pmatrix}
   0 \\
   7 \\
   0 \\
   3 \\
   \end{pmatrix}, \quad
   \begin{pmatrix}
   0 \\
   2 \\
   1 \\
   1 \\
   \end{pmatrix}, \quad
   \begin{pmatrix}
   5 \\
   -5 \\
   0 \\
   7 \\
   \end{pmatrix}.
   \]

4. Let $A = \begin{pmatrix}
   -1 & 1 \\
   1 & 7 \\
   \end{pmatrix}$, and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$. Find the area of $T(S)$, if $S$ is a triangle in $\mathbb{R}^2$ with area 2.

5. Let
   \[
   A = \begin{pmatrix}
   2 & -8 & 6 & 8 \\
   3 & -9 & 5 & 10 \\
   -3 & 0 & 1 & -2 \\
   1 & -4 & 0 & 6 \\
   \end{pmatrix} \quad \text{and} \quad
   B = \begin{pmatrix}
   0 & 1 & 5 & 4 \\
   1 & -1 & -3 & 0 \\
   -1 & 0 & 5 & 4 \\
   3 & -3 & -2 & 5 \\
   \end{pmatrix}
   \]
   a) Compute $\det(A)$.
   b) Compute $\det(B)$.
   c) Compute $\det(AB)$.
   d) Compute $\det(A^2B^{-1}AB^2)$.

6. If $A$ is a $3 \times 3$ matrix and $\det(A) = 1$, what is $\det(-2A)$?

7. a) Is there a real $2 \times 2$ matrix $A$ that satisfies $A^4 = -I_2$? Either write such an $A$, or show that no such $A$ exists.
   (hint: think geometrically! The matrix $-I_2$ represents rotation by $\pi$ radians).
   b) Is there a real $3 \times 3$ matrix $A$ that satisfies $A^4 = -I_3$? Either write such an $A$, or show that no such $A$ exists.
Supplemental problems: §5.1

1. True or false. Answer true if the statement is always true. Otherwise, answer false.
   a) If $A$ and $B$ are $n \times n$ matrices and $A$ is row equivalent to $B$, then $A$ and $B$ have
      the same eigenvalues.
   b) If $A$ is an $n \times n$ matrix and its eigenvectors form a basis for $\mathbb{R}^n$, then $A$ is invertible.
   c) If 0 is an eigenvalue of the $n \times n$ matrix $A$, then rank$(A) < n$.
   d) The diagonal entries of an $n \times n$ matrix $A$ are its eigenvalues.
   e) If $A$ is invertible and 2 is an eigenvalue of $A$, then $\frac{1}{2}$ is an eigenvalue of $A^{-1}$.
   f) If $\det(A) = 0$, then 0 is an eigenvalue of $A$.

2. In this problem, you need not explain your answers; just circle the correct one(s).
   Let $A$ be an $n \times n$ matrix.
   a) Which one of the following statements is correct?
      1. An eigenvector of $A$ is a vector $v$ such that $Av = \lambda v$ for a nonzero scalar $\lambda$.
      2. An eigenvector of $A$ is a nonzero vector $v$ such that $Av = \lambda v$ for a scalar $\lambda$.
      3. An eigenvector of $A$ is a nonzero scalar $\lambda$ such that $Av = \lambda v$ for some vector $v$.
      4. An eigenvector of $A$ is a nonzero vector $v$ such that $Av = \lambda v$ for a nonzero scalar $\lambda$.
   b) Which one of the following statements is not correct?
      1. An eigenvalue of $A$ is a scalar $\lambda$ such that $A - \lambda I$ is not invertible.
      2. An eigenvalue of $A$ is a scalar $\lambda$ such that $(A - \lambda I)v = 0$ has a solution.
      3. An eigenvalue of $A$ is a scalar $\lambda$ such that $Av = \lambda v$ for a nonzero vector $v$.
      4. An eigenvalue of $A$ is a scalar $\lambda$ such that $\det(A - \lambda I) = 0$.

3. Find a basis $B$ for the $(-1)$-eigenspace of $Z = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$

4. Suppose $A$ is an $n \times n$ matrix satisfying $A^2 = 0$. Find all eigenvalues of $A$. Justify your answer.
5. Match the statements (i)-(v) with the corresponding statements (a)-(e). All matrices are $3 \times 3$. There is a unique correspondence. Justify the correspondences in words.

(i) $Ax = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ has a unique solution.

(ii) The transformation $T(v) = Av$ fixes a nonzero vector.

(iii) $A$ is obtained from $B$ by subtracting the third row of $B$ from the first row of $B$.

(iv) The columns of $A$ and $B$ are the same; except that the first, second and third columns of $A$ are respectively the first, third, and second columns of $B$.

(v) The columns of $A$, when added, give the zero vector.

(a) $0$ is an eigenvalue of $A$.
(b) $A$ is invertible.
(c) $\det(A) = \det(B)$
(d) $\det(A) = -\det(B)$
(e) $1$ is an eigenvalue of $A$. 