Please read all instructions carefully before beginning.

- Write your name on the front of each page (not just the cover page!).
- The maximum score on this exam is 50 points, and you have 50 minutes to complete this exam.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit! If you cannot fit your work on the front side of the page, use the back side of the page and indicate that you are using the back side.
- We will hand out loose scrap paper, but it will not be graded under any circumstances. All work must be written on the exam itself.
- You may cite any theorem proved in class or in the sections we covered the text.
- Good luck!
Problem 1.

These problems are true or false. Circle T if the statement is always true. Otherwise, circle F. You do not need to show work or to justify your answer.

a) (T)  F  The augmented matrix below is in RREF.

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & | & 0 \\
0 & 0 & 0 & 1 & | & 1 \\
0 & 0 & 0 & 0 & | & 0
\end{pmatrix}
\]

b) (T)  F  Suppose \( w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, w_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \text{ and } w_3 = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}. \)
Then \( \text{Span}\{w_1, w_2, w_3\} \) is \( \mathbb{R}^3. \)

\[
\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & -1 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}
\]

c) (T)  F  If \( v_1, v_2, v_3 \) are vectors in \( \mathbb{R}^3 \) and the vector equation

\[ x_1 v_1 + x_2 v_2 + x_3 v_3 = b \]

is inconsistent for some \( b \) in \( \mathbb{R}^3 \), then \( \{v_1, v_2, v_3\} \) is linearly dependent.

\[ \text{Span} \{v_1, v_2, v_3\} = \mathbb{R}^3 \]

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(d) (T)  F  Suppose \( A \) is a \( 2 \times 3 \) matrix and \( b \) is a vector so that \( Ax = b \)
is consistent. Then \(-4b\) must be a linear combination of the columns of \( A \).

\( b \) is in column span of \( A \),

Thus \(-4b\) is also in col. span of \( A \).

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e) T  (F)  If \( A \) is an \( m \times n \) matrix and \( n > m \), then the equation \( Ax = b \)
must be inconsistent for some \( b \) in \( \mathbb{R}^m \).

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Columns of \( A \) span \( \mathbb{R}^2 \).
Problem 2.

Short answer. You do not need to show your work except in part (a).

a) (2 points) Let $A = \begin{pmatrix} -2 & 1 \\ 2 & 1 \\ 4 & 0 \end{pmatrix}$ and $x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Compute $Ax$.

$$Ax = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -4 \end{pmatrix}.$$

b) (2 points) Complete the following definition (be mathematically precise!):

Let $w, v_1, v_2, \ldots, v_p$ be vectors in $\mathbb{R}^n$. We say $w$ is a linear combination of $v_1, v_2, \ldots, v_p$ if...

$$w = x_1 v_1 + \cdots + x_p v_p$$

for some scalars $x_1, \ldots, x_p$.

c) (2 points) Suppose $v_1, v_2, v_3$ are vectors in $\mathbb{R}^4$. Which of the following statements must be true? Circle all that apply.

(i) If $\{v_1, v_2, v_3\}$ is linearly dependent, then the vector equation

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$$

has a solution satisfying $x_1 \neq 0$, $x_2 \neq 0$, and $x_3 \neq 0$.

(ii) If $\{v_1, v_2, v_3\}$ is linearly independent, then $\{v_1, v_2\}$ is linearly independent.

d) (2 points) Consider the vectors $a, b$, and $v$ below.

Which of the following describes $v$ in terms of $a$ and $b$?

(i) $v = a + b$  (ii) $v = a - b$  (iii) $v = b - a$  (iv) $v = -a - b$

e) (3 points) Suppose we are given a consistent system of 4 linear equations in 3 variables. Which of the following are possible for the solution set of the system? Circle all that apply.

(i) The system has a unique solution.

(ii) The solution set forms a line in $\mathbb{R}^4$.

(iii) The solution set forms a plane in $\mathbb{R}^3$.
Problem 3.

Show your work in part (a).

a) (4 points)

Is \( \begin{pmatrix} 1 \\ 5 \\ -11 \end{pmatrix} \) in the span of \( \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} \) and \( \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} \)? If your answer is yes, write \( \begin{pmatrix} 1 \\ 5 \\ -11 \end{pmatrix} \) as a linear combination of \( \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} \) and \( \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} \). If your answer is no, justify why.

\[
\begin{pmatrix}
1 & 2 \\
2 & 1 \\
-5 & -4
\end{pmatrix}
\begin{pmatrix}
5 \\
-11
\end{pmatrix}
= \begin{pmatrix}
1 \\
-1
\end{pmatrix}
\begin{pmatrix}
1 \\
-1
\end{pmatrix}
\begin{pmatrix}
3 \\
-1
\end{pmatrix}
\]

\[
\begin{pmatrix}
2 \\
-5
\end{pmatrix}
= 3 \begin{pmatrix}
2 \\
-4
\end{pmatrix}
= \begin{pmatrix}
5 \\
-11
\end{pmatrix}
\]

b) (5 points) Suppose \( A \) is a \( 4 \times 2 \) matrix and the solution set to \( Ax = b \) is drawn below for some vector \( b \).

(i) Draw the solution set to \( Ax = 0 \) on the same graph above.

(ii) Fill the blanks below:

 Geometrically, the span of the columns of \( A \) is a \( \text{line} \) in \( \mathbb{R}^4 \).

\( A \) has \( 1 \) pivot column and \( 1 \) non-pivot column from (a).

\( A \neq b \) and \( 1 \) free variable.
Problem 4.

Part (a) is free response. Show your work or you may receive little or no credit, even if your answer is correct.

a) Snidely Whiplash has given you the following consistent system of equations:

\[ \begin{align*}
   x_1 - x_2 - 4x_3 - x_4 &= 4 \\
   -x_1 + 2x_2 + 8x_3 + 3x_4 &= -5 \\
   2x_1 - x_2 - 4x_3 &= 7.
\end{align*} \]

(i) (5 points) Write the system as an augmented matrix, and put the matrix into reduced row echelon form.

\[
\begin{pmatrix}
   1 & -1 & -4 & 1 \\
   -1 & 2 & 8 & 3 \\
   2 & -1 & -4 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
   1 & -1 & -4 & -1 \\
   0 & 1 & 2 & -1 \\
   0 & 0 & 0 & 0
\end{pmatrix}
\]

(ii) (4 points) Write the solution set to the system of equations in parametric vector form. Clearly indicate which variables (if any) are free.

\[
\begin{pmatrix}
   x_1 \\
   x_2 \\
   x_3 \\
   x_4
\end{pmatrix}
= \begin{pmatrix}
   3 - x_4 \\
   -1 - 4x_3 - 2x_4 \\
   x_3 \\
   x_4
\end{pmatrix}
= \begin{pmatrix}
   3 \\
   -1 \\
   0 \\
   0
\end{pmatrix}
+ x_3 \begin{pmatrix}
   0 \\
   0 \\
   1 \\
   0
\end{pmatrix}
+ x_4 \begin{pmatrix}
   0 \\
   -1 \\
   0 \\
   1
\end{pmatrix}
\]

b) (Unrelated to (a), worth 2 points) Is there a \( 2 \times 2 \) matrix \( A \) so that the solution set to \( Ax = 0 \) is the line \( x_1 - x_2 = 3 \)? If your answer is yes, write such a matrix \( A \). If your answer is no, justify why there is no such matrix \( A \).

\[
A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\[
\Rightarrow x_1 - x_2 = 3
\]

but \( 0 - 0 \neq 3 \). (Line does not pass through origin)
Problem 5.

Parts (a) and (b) are unrelated. Show your work in part (a).

a) (5 points) Let \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) be the vectors below.

\[
\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -2 \\ 8 \\ 6 \end{pmatrix}.
\]

Show that \( \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) is linearly dependent, and find a linear dependence relation for the vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \).

\[
\begin{pmatrix} 0 & 1 & -2 \\ 1 & -2 & 8 \\ 2 & 1 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 8 \\ 0 & 1 & -2 \\ 2 & 1 & 6 \end{pmatrix} \quad \text{R}_1 \leftrightarrow \text{R}_2
\]

\[
\begin{pmatrix} 1 & -2 & 8 \\ 0 & 1 & -2 \\ 0 & 5 & -10 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 8 \\ 0 & 1 & -2 \\ 0 & 5 & -10 \end{pmatrix} \quad \text{R}_3 = \text{R}_3 - 5\text{R}_2
\]

\[
\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{when } x_3 = 1, \text{ get }
\]

\[-4\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = 0.
\]

b) (4 points) Write a matrix \( \mathbf{A} \) so that the solution set to \( \mathbf{A}\mathbf{x} = 0 \) is a line in \( \mathbb{R}^3 \) and the equation \( \mathbf{A}\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \) is consistent.

Many examples possible, for example

\[
\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

\( \mathbf{A} \) must be \( 4 \times 3 \), have two pivots (one free var. for \( \mathbf{A}\mathbf{x} = 0 \)) and have \( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \) in its column span.