1. True or false (justify your answer). Answer true if the statement is always true. Otherwise, answer false.

   a) If $A$ is a $3 \times 4$ matrix and $B$ is a $4 \times 2$ matrix, then the linear transformation $Z$ defined by $Z(x) = ABx$ has domain $\mathbb{R}^3$ and codomain $\mathbb{R}^2$.

   b) If $A$ is an $n \times n$ matrix and the equation $Ax = b$ has at least one solution for each $b$ in $\mathbb{R}^n$, then the solution is unique for each $b$ in $\mathbb{R}^n$.

   c) Suppose $A$ is an $n \times n$ matrix and every vector in $\mathbb{R}^n$ can be written as a linear combination of the columns of $A$. Then $A$ must be invertible.

Solution.

   a) False. In order for $Bx$ to make sense, $x$ must be in $\mathbb{R}^2$, and so $Bx$ is in $\mathbb{R}^4$ and $A(Bx)$ is in $\mathbb{R}^3$. Therefore, the domain of $Z$ is $\mathbb{R}^2$ and the codomain of $Z$ is $\mathbb{R}^3$.

   b) True. The first part says the transformation $T(x) = Ax$ is onto. Since $A$ is $n \times n$, this is the same as saying $A$ is invertible, so $T$ is one-to-one and onto. Therefore, the equation $Ax = b$ has exactly one solution for each $b$ in $\mathbb{R}^n$.

   c) True. If the columns of $A$ span $\mathbb{R}^n$, then $A$ is invertible by the Invertible Matrix Theorem. We can also see this directly without quoting the IMT:

If the columns of $A$ span $\mathbb{R}^n$, then $A$ has $n$ pivots, so $A$ has a pivot in each row and column, hence its matrix transformation $T(x) = Ax$ is one-to-one and onto and thus invertible. Therefore, $A$ is invertible.
2. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be rotation clockwise by $60^\circ$. Let $U : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation satisfying $U(1,0) = (-2, 1)$ and $U(0,1) = (1, 0)$.

a) Find the standard matrix for the composition $U \circ T$ using matrix multiplication.

b) Find the standard matrix for the composition $T \circ U$ using matrix multiplication.

c) Is rotating clockwise by $60^\circ$ and then performing $U$, the same as first performing $U$ and then rotating clockwise by $60^\circ$?

Solution.

a) The matrix for $T$ is $\begin{pmatrix} \cos(-60^\circ) & -\sin(-60^\circ) \\ \sin(-60^\circ) & \cos(-60^\circ) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$.

The matrix for $U$ is $\begin{pmatrix} U(e_1) & U(e_2) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$.

The matrix for $U \circ T$ is $\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 - \frac{\sqrt{3}}{2} & 1 - \frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{1}{2} + \sqrt{3} \end{pmatrix}$.

b) The matrix for $T \circ U$ is $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 + \frac{\sqrt{3}}{2} & \frac{1}{2} - \frac{\sqrt{3}}{2} \\ \frac{1}{2} + \sqrt{3} & -\frac{\sqrt{3}}{2} \end{pmatrix}$.

c) No. In (a) and (b), we found that the standard matrices for $U \circ T$ and $T \circ U$ are different, so the transformations are different.