Math 1553 Worksheet §§5.1, 5.2, 5.4

1. Answer yes, no, or maybe. Justify your answers. In each case, $A$ is a matrix whose entries are real numbers.

a) Suppose $A = \begin{pmatrix} 3 & 0 & 0 \\ 5 & 1 & 0 \\ -10 & 4 & 7 \end{pmatrix}$. Then the characteristic polynomial of $A$ is $\det(A - \lambda I) = (3 - \lambda)(1 - \lambda)(7 - \lambda)$.

b) If $A$ is a $3 \times 3$ matrix with characteristic polynomial $-\lambda(\lambda - 5)^2$, then the $5$-eigenspace is 2-dimensional.

c) If $A$ is an invertible $2 \times 2$ matrix, then $A$ is diagonalizable.

Solution.

a) Yes. Since $A - \lambda I$ is triangular, its determinant is the product of its diagonal entries.

b) Maybe. The geometric multiplicity of $\lambda = 5$ can be 1 or 2. For example, the matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 2-dimensional, whereas the matrix $\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 1-dimensional. Both matrices have characteristic polynomial $-\lambda(\lambda - 5)^2$.

c) Maybe. The identity matrix is invertible and diagonalizable, but the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible but not diagonalizable.
2. The eigenspaces of some $2 \times 2$ matrix $A$ are drawn below. Write an invertible matrix $C$ and a diagonal matrix $D$ so that $A = CDC^{-1}$.

**Solution:** We choose $D$ to be a diagonal matrix whose entries are the eigenvalues of $A$, and $C$ a matrix whose columns are corresponding eigenvectors (written in the same order).

The eigenvalues of $A$ are $\lambda_1 = -1$ and $\lambda_2 = -2$.

The ($-1$)-eigenspace is spanned by $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The ($-2$)-eigenspace is spanned by $v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Therefore, we can choose $C = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$.

There are other possibilities for $C$ and $D$.

For example, since $\text{Span} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} = \text{Span} \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$, we could have chosen $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ instead. Regardless, if you write any correct answers for $C$ and $D$ and go the extra step of carrying out the computation, you will obtain

$$A = CDC^{-1} = \frac{-1}{5} \begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}.$$
3. Let

\[ A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1}. \]

Find a formula for \( A^n \) (where \( n \) is a positive integer).

**Solution:** The matrix \( A \) has already been diagonalized for us as \( A = CDC^{-1} \) for the matrices above. We find \( C^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \) so

\[ A^n = CD^nC^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2^n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2^n} & \frac{1}{2^n} \end{pmatrix} \]

\[ = \begin{pmatrix} 1 & 0 \\ 1 - \frac{1}{2^n} & \frac{1}{2^n} \end{pmatrix}. \]