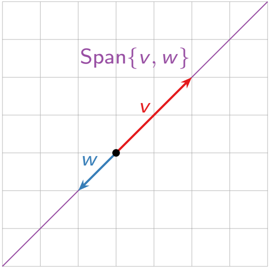
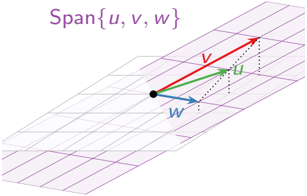
Section 2.5

Linear Independence

Motivation

Sometimes the span of a set of vectors is "smaller" than you expect from the number of vectors.





This means that (at least) one of the vectors is *redundant*: you're using "too many" vectors to describe the span.

Notice in each case that one vector in the set is already in the span of the others—so it doesn't make the span bigger.

Today we will formalize this idea in the concept of *linear (in)dependence*.

Linear Independence

Definition

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$. The set $\{v_1, v_2, \ldots, v_p\}$ is **linearly dependent** otherwise.

In other words, $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent if there exist numbers $x_1, x_2, \ldots, x_p$, not all equal to zero, such that

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0.$$

This is called a **linear dependence relation** or an **equation of linear dependence**.

Like span, linear (in)dependence is another one of those big vocabulary words that you absolutely need to learn. Much of the rest of the course will be built on these concepts, and you need to know exactly what they mean in order to be able to answer questions on quizzes and exams (and solve real-world problems later on).

Definition

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ in $\mathbf{R}^n$ is **linearly independent** if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$. The set $\{v_1, v_2, \ldots, v_p\}$ is **linearly dependent** otherwise.

Note that linear (in)dependence is a notion that applies to a *collection of vectors*, not to a single vector, or to one vector in the presence of some others.

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}3\\1\\4\end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

$$x\begin{pmatrix}1\\1\\1\end{pmatrix} + y\begin{pmatrix}1\\-1\\2\end{pmatrix} + z\begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

have a nontrivial solution? How do we solve this kind of vector equation?

$$\begin{pmatrix}1 & 1 & 3\\1 & -1 & 1\\1 & 2 & 4\end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 0\end{pmatrix}$$

So $x = -2z$ and $y = -z$. So the vectors are linearly *dependent*, and an equation of linear dependence is (taking $z = 1$)

$$-2\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\-1\\2\end{pmatrix} + \begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

[interactive]

Checking Linear Independence

Question: Is $\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \right\}$ linearly independent?

Equivalently, does the (homogeneous) the vector equation

$$x \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

have a nontrivial solution?

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -2 & 2 & 4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The trivial solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is the unique solution. So the vectors are linearly *in*dependent.

[interactive]

Linear Independence and Matrix Columns

In general, $\{v_1, v_2, \ldots, v_p\}$ is linearly independent if and only if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution, if and only if the matrix equation

$$Ax = 0$$

has only the trivial solution, where A is the matrix with columns $v_1, v_2, \ldots, v_p$:

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

This is true if and only if the matrix A has a pivot in each column.

Important

- The vectors $v_1, v_2, \ldots, v_p$ are linearly independent if and only if the matrix with columns $v_1, v_2, \ldots, v_p$ has a pivot in each column.
- Solving the matrix equation $Ax = 0$ will either verify that the columns $v_1, v_2, \ldots, v_p$ of A are linearly independent, or will produce a linear dependence relation.

Suppose that one of the vectors $\{v_1, v_2, \dots, v_p\}$ is a linear combination of the other ones (that is, it is in the span of the other ones):

$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$$

Then the vectors are linearly *de*pendent:

$$2v_1 - \frac{1}{2}v_2 - v_3 + 6v_4 = 0.$$

Conversely, if the vectors are linearly dependent

$$2v_1 - \frac{1}{2}v_2 + 6v_4 = 0.$$

then one vector is a linear combination of (in the span of) the other ones:

$$v_2 = 4v_1 + 12v_4.$$

Theorem

A set of vectors $\{v_1, v_2, \dots, v_p\}$ is linearly *de*pendent if and only if one of the vectors is in the span of the other ones.

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Equivalently:

Theorem

A set of vectors $\{v_1, v_2, \dots, v_p\}$ is linearly *de*pendent if and only if you can remove one of the vectors without shrinking the span.

Indeed, if $v_2 = 4v_1 + 12v_3$, then a linear combination of v_1, v_2, v_3 is

$$\begin{aligned} x_1 v_1 + x_2 v_2 + x_3 v_3 &= x_1 v_1 + x_2(4v_1 + 12v_3) + x_3 v_3 \\ &= (x_1 + 4x_2)v_1 + (12x_2 + x_3)v_3, \end{aligned}$$

which is already in $\mathrm{Span}\{v_1, v_3\}$.

Conclusion: v_2 was redundant.

Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly *de*pendent if and only if one of the vectors is in the span of the other ones.

Better Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent if and only if there is some j such that v_j is in $\text{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

Equivalently, $\{v_1, v_2, \ldots, v_p\}$ is linearly *in*dependent if for every j, the vector v_j is not in $\text{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

This means $\text{Span}\{v_1, v_2, \ldots, v_j\}$ is *bigger* than $\text{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

Translation

A set of vectors is linearly independent if and only if, every time you add another vector to the set, the span gets bigger.

Better Theorem

A set of vectors $\{v_1, v_2, \ldots, v_p\}$ is linearly dependent if and only if there is some j such that v_j is in $\text{Span}\{v_1, v_2, \ldots, v_{j-1}\}$.

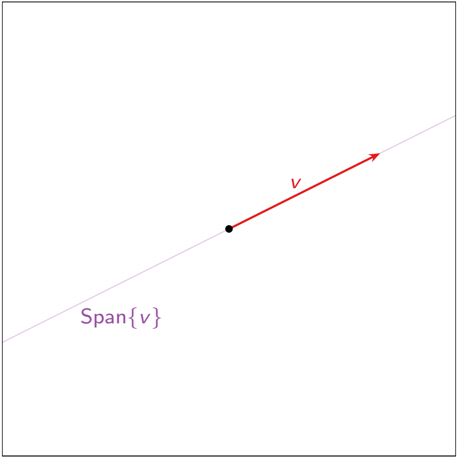
Why? Take the largest j such that v_j is in the span of the others. Then v_j is in the span of $v_1, v_2, \ldots, v_{j-1}$. Why? If not ($j = 3$):

$$v_3 = 2v_1 - \frac{1}{2}v_2 + 6v_4$$

Rearrange:

$$v_4 = -\frac{1}{6}\left(2v_1 - \frac{1}{2}v_2 - v_3\right)$$

so v_4 works as well, but v_3 was supposed to be the last one that was in the span of the others.

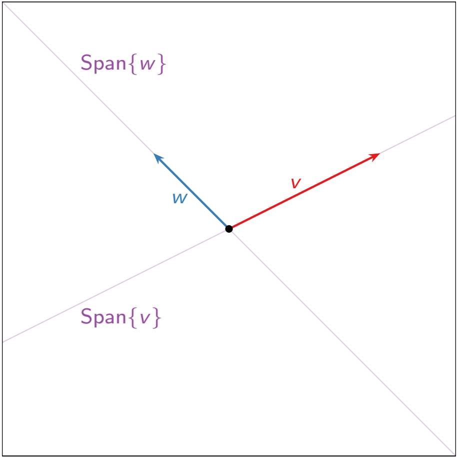


[interactive 2D: 2 vectors]

[interactive 2D: 3 vectors]

One vector $\{v\}$:

Linearly independent if $v \neq 0$.

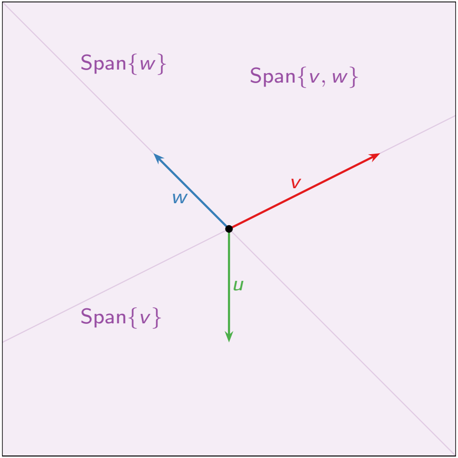


[interactive 2D: 2 vectors]
[interactive 2D: 3 vectors]

One vector $\{v\}$:
Linearly independent if $v \neq 0$.

Two vectors $\{v, w\}$:
Linearly independent

- Neither is in the span of the other.
- Span got bigger.



[interactive 2D: 2 vectors]

[interactive 2D: 3 vectors]

One vector $\{v\}$:
Linearly independent if $v \neq 0$.

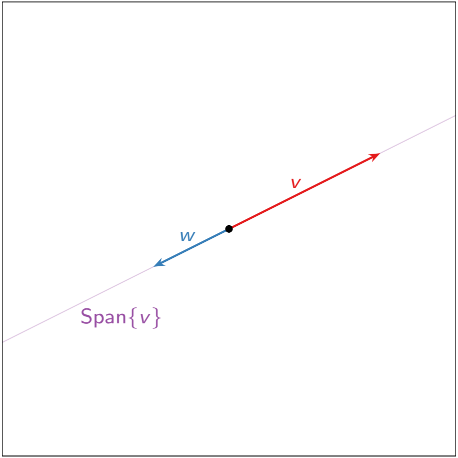
Two vectors $\{v, w\}$:
Linearly independent

- Neither is in the span of the other.
- Span got bigger.

Three vectors $\{v, w, u\}$:
Linearly dependent:

- u is in Span$\{v, w\}$.
- Span didn't get bigger after adding u.
- Can remove u without shrinking the span.

Also v is in Span$\{u, w\}$ and w is in Span$\{u, v\}$.

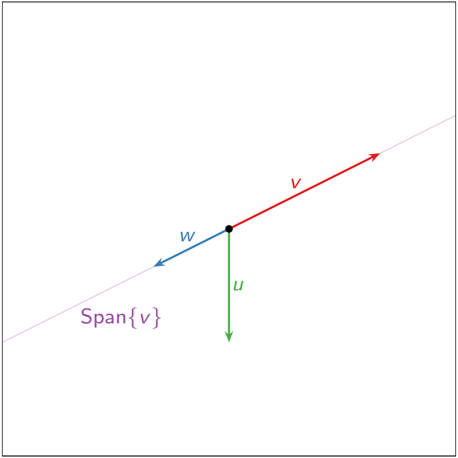


[interactive 2D: 2 vectors]
[interactive 2D: 3 vectors]

Two collinear vectors $\{v, w\}$:
Linearly dependent:

- w is in Span$\{v\}$.
- Can remove w without shrinking the span.
- Span didn't get bigger when we added w.

Observe: *Two* vectors are linearly *dependent* if and only if they are *collinear*.

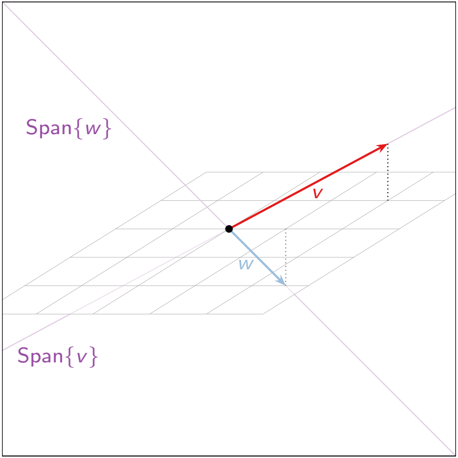


[interactive 2D: 2 vectors]
[interactive 2D: 3 vectors]

Three vectors $\{v, w, u\}$:
Linearly dependent:

- w is in Span$\{u, v\}$.
- Can remove w without shrinking the span.
- Span didn't get bigger when we added w.

Observe: If a set of vectors is linearly dependent, then so is any larger set of vectors!



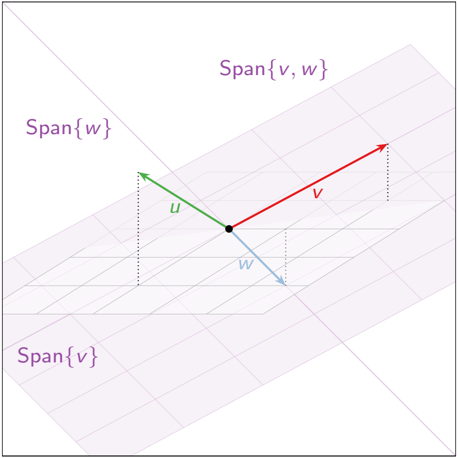
[interactive 3D: 2 vectors]

[interactive 3D: 3 vectors]

Two vectors $\{v, w\}$:

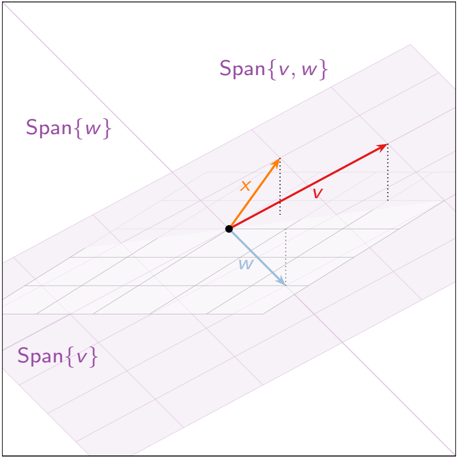
Linearly independent:

- Neither is in the span of the other.
- Span got bigger when we added w.



[interactive 3D: 2 vectors]
[interactive 3D: 3 vectors]

Three vectors $\{v, w, u\}$:
Linearly independent: span got bigger when we added u.



[interactive 3D: 2 vectors]

[interactive 3D: 3 vectors]

Three vectors $\{v, w, x\}$:
Linearly dependent:

- x is in Span$\{v, w\}$.
- Can remove x without shrinking the span.
- Span didn't get bigger when we added x.

Poll

Poll

Are there four vectors u, v, w, x in $\mathbf{R}^3$ which are linearly dependent, but such that u is *not* a linear combination of v, w, x? If so, draw a picture; if not, give an argument.

Yes: actually the pictures on the previous slides provide such an example.

Linear dependence of $\{v_1, \ldots, v_p\}$ means *some* v_i is a linear combination of the others, not *any*.

Linear Dependence and Free Variables

Theorem

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbf{R}^n$, and consider the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

Then you can delete the columns of A *without* pivots (the columns corresponding to free variables), without changing $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$. The pivot columns are linearly independent, so you can't delete any more columns.

This means that each time you add a pivot column, then the span increases.

Upshot

Let d be the number of pivot columns in the matrix A above.

- If $d = 1$ then $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is a line.
- If $d = 2$ then $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is a plane.
- If $d = 3$ then $\mathrm{Span}\{v_1, v_2, \ldots, v_p\}$ is a 3-space.
- Etc.

Why? If the matrix is in RREF:

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then the column without a pivot is in the span of the pivot columns:

$$\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and the pivot columns are linearly independent:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_4 \end{pmatrix} \implies x_1 = x_2 = x_4 = 0.$$

Why? If the matrix is not in RREF, then row reduce:

$$A = \begin{pmatrix} 1 & 7 & 23 & 3 \\ 2 & 4 & 16 & 0 \\ -1 & -2 & -8 & 4 \end{pmatrix} \quad \xrightarrow{\text{RREF}} \quad \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The following vector equations have the same solution set:

$$x_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 7 \\ 4 \\ -2 \end{pmatrix} + x_3 \begin{pmatrix} 23 \\ 16 \\ -8 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = 0$$

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

We conclude that $\begin{pmatrix} 23 \\ 16 \\ -8 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + 3 \begin{pmatrix} 7 \\ 4 \\ -2 \end{pmatrix} + 0 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

and that $x_1 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 7 \\ 4 \\ -2 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = 0$ has only the trivial solution.

Fact 1: Say $v_1, v_2, \ldots, v_n$ are in $\mathbf{R}^m$. If $n > m$ then $\{v_1, v_2, \ldots, v_n\}$ is linearly *dependent*: the matrix

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

cannot have a pivot in each column (it is too wide).

This says you can't have 4 linearly independent vectors in $\mathbf{R}^3$, for instance.

> A wide matrix can't have linearly independent columns.

Fact 2: If one of $v_1, v_2, \ldots, v_n$ is zero, then $\{v_1, v_2, \ldots, v_n\}$ is linearly *dependent*. For instance, if $v_1 = 0$, then

$$1 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3 + \cdots + 0 \cdot v_n = 0$$

is a linear dependence relation.

> A set containing the zero vector is linearly dependent.

Summary

- A set of vectors is **linearly independent** if removing one of the vectors shrinks the span; otherwise it's **linearly dependent**.
- There are several other criteria for linear (in)dependence which lead to pretty pictures.
- The columns of a matrix are linearly independent if and only if the RREF of the matrix has a pivot in every *column*.
- The pivot columns of a matrix A are linearly independent, and you can delete the non-pivot columns (the "free" columns) without changing the span of the columns.
- Wide matrices cannot have linearly independent columns.

Warning

These are not the official definitions!