Practice Midterm 3, Solutions

Solutions

1. The vector from $(1,0)$ to $(4,5)$ is $(3,5)$ and the vector from $(1,0)$ to $(1,-4)$ is $(0,-4)$. So the area of the triangle is

$$\frac{1}{2}\left|\det\begin{pmatrix} 3 & 0 \\ 5 & -4 \end{pmatrix}\right| = \frac{1}{2}(12) = 6.$$

2. (a) True, and in fact T is its own inverse.
(b) True. A variety of ways to see this. One way is to write the matrix A for T and note that it has a 3×3 matrix with 3 pivots, therefore A is invertible and T is invertible by the Invertible Matrix Theorem.
(c) True: The identity transformation is invertible.

3. We solve

$$\det\begin{pmatrix} 1 & 0 & 4 \\ 0 & c & -5 \\ 1 & 3 & 7 \end{pmatrix} = 3$$

$$7c + 15 - 4c = 3, \qquad 3c = -12, \qquad c = -4.$$

4. A is 3×3 and $\det(A) = 4$, so

$$\det(-2A^{-1}) = (-2)^3 \det(A^{-1}) = (-8)(1/4) = -2.$$

5. We are told that A is 5×5 and $\det(A) = 3$.

a) True. The columns of A form a basis for $\mathbf{R}^n$, since A is invertible.

b) True. The columns of A are linearly independent since A is invertible.

c) False. The rank of A is 5 since A is invertible.

d) True. The null space of A is just the zero vector, since $Ax = 0$ has only the trivial solution.

6. This problem comes from the 5.1 Supplement.

a) The correct answer is (III).

b) The correct answer is (III).

7. **a)** Since A has $\lambda = -1$ as an eigenvalue, the equation $(A + I)x = 0$ has infinitely many solutions since $Ax = -x$ has a non-trivial solution.

b) $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 3$, and to get the matrix below requires a row swap and multiplying a row by -2, so

$$\det\begin{pmatrix} -2c & -2d \\ a & b \end{pmatrix} = 3(-1)(-2) = 6.$$

8. $A = \begin{pmatrix} 7 & 4 & 4 \\ 4 & 7 & 4 \\ 0 & 0 & 4 \end{pmatrix}$ so

$$(A - 3I|0) = \left(\begin{array}{ccc|c} 4 & 4 & 4 & 0 \\ 4 & 4 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This gives $x_1 + x_2 = 0$, x_2 free, and $x_3 = 0$, so a basis for the 3-eigenspace is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

9. **a)** True. The matrix A gives counterclockwise rotation by 23°, which means that if $v \neq 0$, then v and Av will not be on the same line through the origin. Therefore, A doesn't have any real eigenvalues.

b) True: u and v are eigenvectors for $\lambda = 2$ and $u + v$ is not the zero vector, so $u + v$ is also a 2-eigenvector. You can see this by recalling that the 2-eigenspace is a subspace (thus closed under addition), or note

$$A(u + v) = Au + Av = 2u + 2v = 2(u + v).$$

10. Since $A = \begin{pmatrix} 1 & k \\ 1 & 3 \end{pmatrix}$, so its char. polynomial is

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 4\lambda + 3 - k.$$

This has one real eigenvalue of algebraic multiplicity 2 precisely when the polyomial is a square, so it equals

$$(\lambda - 2)^2 = \lambda^2 - 4\lambda + 4,$$

thus $3 - k = 4$ so $k = -1$.

11. We expand the characteristic polynomial along the third row: $A = \begin{pmatrix} 1 & 4 & -1 \\ 2 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ so

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 4 & -1 \\ 2 & 3-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (-1)^6 (1-\lambda)\big[(1-\lambda)(3-\lambda) - 8\big]$$
$$= (1-\lambda)(\lambda^2 - 4\lambda - 5) = (1-\lambda)(\lambda - 5)(\lambda + 1).$$

The eigenvalues are $\lambda = -1$, $\lambda = 1$, $\lambda = 5$.

12. $A = \begin{pmatrix} 3 & -7 \\ 1 & -2 \end{pmatrix}$ so

$$A^{-1} = \frac{1}{3(-2) - (-7)(1)} \begin{pmatrix} -2 & 7 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 7 \\ -1 & 3 \end{pmatrix}.$$

13. **a)** $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is not diagonalizable.

Its only eigenvalue is $\lambda = 1$, but $\mathrm{Nul}(A - I)$ gives only two free variables, so the 1-eigenspace only has dimension 2.

b) Yes, B is a 2×2 matrix with two real eigenvalues $\lambda = 1$ and $\lambda = -1$, so B is diagonalizable.

14. Since $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is in the 1-eigenspace and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is in the 2-eigenspace, we get

$$A\left(\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) = A\begin{pmatrix} 4 \\ 1 \end{pmatrix} + A\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}.$$

So $k = 5$.

15. We are told the 2×2 matrix A has eigenvalue $\lambda_1 = -2 + i\sqrt{5}$ and corresponding eigenvector $\begin{pmatrix} 10 \\ -5 - i\sqrt{5} \end{pmatrix}$.

a) Complex eigenvalues come in complex conjugate pairs, so $\lambda_2 = -2 - i\sqrt{5}$ is its other eigenvalue.

b) We get an eigenvector for $\lambda = 2$ by taking the complex conjugate of each entry of the eigenvector for λ_1, which gives us $\begin{pmatrix} 10 \\ -5 + i\sqrt{5} \end{pmatrix}$.

16. **a)** True. If $A = CDC^{-1}$ and A is invertible then so are all three matrices on the right side of the equation, and

$$A^{-1} = (CDC^{-1})^{-1} = (C^{-1})^{-1}D^{-1}C^{-1} = CD^{-1}C^{-1}.$$

b) True:

$$\det(A) = \det(CDC^{-1}) = \det(C)\det(D)\det(C^{-1}) = \det(C)\det(D)\frac{1}{\det(C)} = \det(D).$$

17. The matrix for T is $A = \begin{pmatrix} k & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & k^3 \end{pmatrix}$, so the volume of $T(S)$ is

$$\det(A)\mathrm{Vol}(S) = k^6(2021) = 2021k^6.$$

18. Here, A is the 2×2 matrix whose 2-eigenspace is the line $x_2 = 3x_1$ and whose null space is the line $x_2 = -x_1$. Therefore, the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 0$, and corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Therefore, by the Diagonalization Theorem we have $A = CDC^{-1}$ where

$$C = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

19. **a)** True. If A is 7×7 then it must have at least one real eigenvalue. Since (non-real) complex eigenvalues (and their powers) come in conjugate pairs, only an “even” × “even” matrix A can have no real eigenvalues.

Alternatively: since $\det(A - \lambda I)$ is a degree 7 polynomial, it has at least one real root just due to a precalculus argument using end-behavior and continuity of polynomial functions.

b) True. If A is 4×4 and if i and $3i$ are eigenvalues of A, then so are $-i$ and $-3i$, so none of the four eigenvalues of A are real numbers.

20. Here, $A = \begin{pmatrix} 3 & c \\ 2 & 1 \end{pmatrix}$ and we need $\lambda = 2$ to be an eigenvalue. This is the same as $A - 2I$ is not invertible. We row-reduce

$$(A - 2I \,|\, 0) = \left(\begin{array}{cc|c} 1 & c & 0 \\ 2 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & c & 0 \\ 0 & -1-2c & 0 \end{array}\right)$$

Since $A - 2I$ is not invertible, we have $-1 - 2c = 0$, so $c = -1/2$. Alternatively, we could have solved for $\det(A - 2I) = 0$ and found $c = -1/2$.