Midterm 3 (3.5-5.5)

1. Compute the inverse of $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$.

   (a) $\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$

   (d) $\begin{pmatrix} -3 & 1 \\ 2 & -1 \end{pmatrix}$

   (e) $\begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}$

   (f) $\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$

   Solution: This is a quintessential inverse problem, basically #1 from the 3.5-3.6 Webwork. The inverse is $\frac{1}{3(1)-2(1)}\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$, which is answer (f).

2. Answer yes, no, or maybe to each of the following questions. In each case, A is a matrix whose entries are real numbers.

   (a) Suppose that A is a 3×3 matrix whose 1-eigenspace is a line and whose 2-eigenspace is a plane. Is A invertible?

   (b) Suppose that A is a 5×5 matrix and that the set of solutions to $Ax = e_5$ is a line in $\mathbf{R}^5$. Is A invertible?

   Solution:

   (a) Yes. From the information given, the characteristic polynomial of A must be $(1-\lambda)(2-\lambda)^2$, so $\lambda = 0$ is not an eigenvalue of A and therefore A is invertible.

   (b) No. We are given that $Ax = e_5$ has infinitely many solutions, thus $Ax = 0$ has infinitely many solutions and A is not invertible by the Invertible Matrix Theorem.

3. Suppose A and B are the invertible 2×2 matrices whose inverses satisfy

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}.$$

   Find $(AB)^{-1}$.

   (a) $\begin{pmatrix} 1 & -3 \\ 2 & 6 \end{pmatrix}$

   (b) $\begin{pmatrix} 1/2 & 1/4 \\ -1/6 & 1/12 \end{pmatrix}$

   (c) $\begin{pmatrix} 5/12 & 1/6 \\ -1/12 & 1/6 \end{pmatrix}$

   (d) $\begin{pmatrix} 2 & -2 \\ 1 & 5 \end{pmatrix}$

(e) $\begin{pmatrix} 5 & -2 \\ -1 & 2 \end{pmatrix}$

Solution: The answer is (d), using the key fact $(AB)^{-1} = B^{-1}A^{-1}$.

$$(AB)^{-1} = B^{-1}A^{-1} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & 5 \end{pmatrix}$$

4. Suppose $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 3$. Find the determinant of the matrix below.

$$\begin{pmatrix} d & e & f \\ 4a & 4b & 4c \\ g-2a & h-2b & i-2c \end{pmatrix}$$

(a) 3
(b) -3
(c) 6
(d) -6
(e) 12
(f) -12
(g) 24
(h) -24

Solution: This one was basically taken from the Determinants I Webwork #7 and is also similar to a Quiz 6 and sample midterm problem.

To get the second matrix from the first, we switch the first two rows (multiplying the determinant by -1), then subtract two times row 2 from row 3 (doesn't change determinant), then multiply the second row by 4 (multiplying the determinant by 4).

Putting this together: our final answer is $3(-1)(4) = -12$.

5. Say that R is a rectangle in $\mathbf{R}^2$ with side lengths 3 and 4, and that $T(v) = Av$ is the matrix transformation where

$$A = \begin{pmatrix} 2 & 9 \\ 1 & 3 \end{pmatrix}.$$

What is the area of $T(R)$?

Solution: This is a slight modification of #9 from the Determinants I Webwork. The area of $T(R)$ is

$$\text{Area}(T(R)) = |\det(A)|\text{Area}(R) = |(6-9)|(3 \cdot 4) = 36.$$

6. Find the value of c so that

$$\det \begin{pmatrix} -1 & 1 & -2 \\ 0 & c & 1 \\ 1 & 3 & 1 \end{pmatrix} = 1.$$

Solution: This problem was taken from problem #3 in the sample midterm, with some numbers changed.

Expanding the determinant using the cofactor expansion along the 2nd row gives us

$$c(-1)^4 \det \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} + 1(-1)^5 \det \begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix} = 1,$$

$$c(-1+2) - (-3-1) = 1$$

$$c + 4 = 1, \qquad c = -3.$$

7. (a) Consider the line L in $\mathbf{R}^2$ given by the equation $y = 7x$, and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors across L. What are the eigenvalues of the standard matrix for T?

   (b) Let A be the 3×3 matrix for the natural projection onto the xy-plane in $\mathbf{R}^3$. What are the eigenvalues of A?

   Hint: It is not necessary to find the standard matrix in (a) or (b) to answer these questions.

Solution: This problem was taken problem #2 from the 5.1-5.2 worksheet with almost no modification.

(a) The eigenvalues are -1 and 1. T fixes vectors along the line $y = 7x$ (so $\lambda = 1$ is an eigenvalue) and flips vectors that are on the perpendicular line $y = -\dfrac{1}{7}x$ (so $\lambda = 1$ is an eigenvalue). The most eigenvalues a 2×2 matrix can have is 2, so the eigenvalues $\lambda = 1$ and $\lambda = -1$ are the eigenvalues of A.

(b) The eigenvalues are 0 and 1. $Av = v$ for all vectors in the xy-plane of $\mathbf{R}^3$ (so $\lambda = 1$ is an eigenvalue) and $Av = 0$ for all vectors on the z-axis (so $\lambda = 0$ is an eigenvalue). By the previous sentence, the geometric multiplicities of $\lambda = 0$ and $\lambda = 1$ sum to 3, so there are no more possible eigenvalues for this 3×3 matrix.

8. Let A be the 2×2 matrix for counterclockwise rotation by $90°$ in $\mathbf{R}^2$. What are the eigenvalues for A?

   (a) -1 and 1
   (b) 1 only
   (c) -1 only
   (d) $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$
   (e) $\dfrac{\pi}{2}$ only
   (f) $-\dfrac{\pi}{2}$ only
   (g) i and $-i$.
   (h) $i\frac{\pi}{2}$ and $-i\frac{\pi}{2}$

Solution: The matrix is $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and we solve for the eigenvalues:

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1,$$

so $\lambda^2 = -1$, thus $\lambda = \pm i$.

9. The number $\lambda = 5$ is an eigenvalue of the matrix $A = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$. Find the value of h so that

$$A \begin{pmatrix} -1 \\ h \end{pmatrix} = 5 \begin{pmatrix} -1 \\ h \end{pmatrix}.$$

Solution: The answer is $h = 1$.

$$(A - 5I \mid 0) = \left(\begin{array}{cc|c} -3 & -3 & 0 \\ -3 & -3 & 0 \end{array} \right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right),$$

so $x_1 + x_2 = 0$. This gives us $x_1 = -x_2$ and x_2 is free, and in parametric vector form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

10. Answer true or false to each of the following questions. In each case, A is a matrix whose entries are real numbers.

    (a) Suppose that A is an $n \times n$ matrix and v is a nonzero vector in the null space of A. Then v is an eigenvector for A.

    (b) Suppose that A is an $n \times n$ matrix and that u and v are eigenvectors of A. Then $u + v$ must be an eigenvector of A.

    Solution:

    (a) True. This emphasizes the fundamental fact that $\lambda = 0$ is an eigenvalue of A if and only if $Av = 0$ for some nonzero vector v, in which case v is an eigenvector corresponding to $\lambda = 0$.

    (b) False. For example, take $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Then $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors of A but $u + v$ is not an eigenvector:

    $$A(u + v) = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

    (The answer would have been "true" if u and v were further assumed to be different vectors in the *same* eigenspace)

11. Suppose A is a 4×4 matrix with characteristic polynomial

    $$\det(A - \lambda I) = (5 - \lambda)(-5 - \lambda)^3.$$

    Which of the following are possible for the dimension of the (-5)-eigenspace? Select all that apply.

    (a) 0

    (b) 1

    (c) 2

    (d) 3

    (e) 4

    Solution: Since $\lambda = -5$ is an eigenvalue and has algebraic multiplicity 3, we know

    $$3 \ \geq \ (\text{geom. mult. of } \lambda = -5) \ \geq \ 1.$$

    So (b), (c), and (d) are possible, but (a) and (e) are impossible.

12. Find all values of k so that the matrix $A = \begin{pmatrix} -2 & k \\ 12 & 10 \end{pmatrix}$ has exactly one real eigenvalue with algebraic multiplicity 2.

    Solution: This is #4 from the 5.2 Webwork with some numbers changed. The characteristic polynomial is

    $$\lambda^2 - \text{Tr}(A)\lambda + \det(A) = \lambda^2 - 8\lambda + (-20 + 12k).$$

    In order for this to be a perfect square, we need it to equal

    $$(\lambda - 4)^2 = \lambda^2 - 8\lambda + 16,$$

    so $16 = -20 + 12k$, and we find $k = -3$.

13. Find a basis for the (-2)-eigenspace of the matrix A below:

$$A = \begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix}.$$

(a) $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

(c) $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$

(d) $\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$

(e) $\left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$

(f) $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$

Solution: This is a standard kind of problem we do often in chapter 5 (for a similar exercise, see #2 from the 5.2 Webwork).

$$(A + 2I \mid 0) = \left(\begin{array}{cc|c} -1 & 2 & 0 \\ -2 & 4 & 0 \end{array} \right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array} \right),$$

so $x_1 - 2x_2 = 0$. Thus $x_1 = 2x_2$ and x_2 is free. Therefore, the (-2)-eigenspace is spanned by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

14. Answer true or false to each of the following questions. In each case, A is a matrix whose entries are real numbers.

(a) Suppose that A is a 5×5 matrix with eigenvalues 6, 7, 8, and 9, and that the 7-eigenspace for A is a two-dimensional plane. Then A must be diagonalizable.

(b) Suppose A is an $n \times n$ matrix and $\lambda = 6$ is an eigenvalue of A. Then the 6-eigenspace of A must be a subspace of $\mathbf{R}^n$.

Solution:

(a) True: A is a 5×5 matrix and the sum of geometric multiplicities of its real eigenvalues is 5, so A is diagonalizable.

(b) True. There are many ways to see this fundamental fact. One way is to note that the 6-eigenspace of A is $\mathrm{Nul}(A - 6I)$, and the null space of an $n \times n$ matrix is automatically a subspace of $\mathbf{R}^n$.

15. Let A be the matrix which has the diagonalization below:

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 4 & 1 \\ -2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 & -2 \\ 0 & 4 & 1 \\ -2 & 0 & 3 \end{pmatrix}^{-1}.$$

Answer the following questions.

(a) Which of the following is a basis for the 2-eigenspace of A?

i. $\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right\}$

ii. $\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\}$

iii. $\left\{ \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \right\}$

iv. $\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \right\}$

(b) Find $A^{35} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$.

(i) $\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$ (ii) $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ (iii) $\begin{pmatrix} (-2)^{35} \\ 1^{35} \\ 3^{25} \end{pmatrix}$ (iv) $\begin{pmatrix} -2 \cdot 2^{35} \\ 2^{35} \\ 3 \cdot 2^{35} \end{pmatrix}$

Solution: This problem is similar to many diagonalization problems from the 5.4 Webwork, class, and #2 from the 5.4-5.5 worksheet,, except that it has saved us from doing most of the work by diagonalizing the matrix for us. We have been given a diagonalization of A, so $A = CDC^{-1}$ where C is a matrix whose columns are eigenvectors of A and D is the diagonal matrix of corresponding vectors (written in matching order!).

(a) $\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\}$. From the diagonalization of A, we see that the first two columns of C are a basis of the 2-eigenspace of A.

(b) $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ is an eigenvector in the (-1)-eigenspace of A, so

$$A^{35} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = (-1)^{35} \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = - \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}.$$

16. Which of the following matrices are diagonalizable? Select all that apply.

(a) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Solution:

(a) Yes: the zero matrix is diagonalizable (in fact, diagonal!).

(b) No: A has exactly only one eigenvalue $\lambda = 4$ with algebraic multiplicity 2, but $A - 4I$ is $\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}$, so the 4-eigenspace only has geometric multiplicity 1.

(c) Yes: the 2×2 matrix has two distinct real eigenvalues and is therefore diagonalizable.

(d) Yes: the characteristic polynomial is $\lambda^2 - 1$, so it has $\lambda = \pm 1$ as eigenvalues and is diagonalizable by the same reasoning as (c).

17. Answer true or false to the following questions. In each case, A is a matrix whose entries are real numbers.

(a) If A is an $n \times n$ diagonal matrix, then A must be diagonalizable.

(b) Suppose A is an $n \times n$ matrix and $\lambda = 2$ is an eigenvalue of A. Then there are infinitely many vectors v in $\mathbf{R}^n$ that satisfy $Av = 2v$.

Solution:

(a) True: $A = IAI^{-1}$.

(b) True: Since 2 is an eigenvalue we know that $(A-2I)v = 0$ has a non-trivial solution and therefore has infinitely many solutions, so $Av = 2v$ has infinitely many solutions.

18. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & h \\ 0 & 0 & h \end{pmatrix}$. For which of the following values of h is A diagonalizable? Select all that apply.

(a) $h = 0$

(b) $h = 1$

(c) $h = -1$

(d) $h = 2$

Solution:

(a) Yes. When $h = 0$, the matrix is diagonal.

(b) No. When $h = 1$, the 1-eigenspace has algebraic multiplicity 2 but geometric multiplicity 1.

(c) Yes. When $h = -1$, we see from the triangular form of A that A is a 3×3 matrix with 3 different real eigenvalues $2, 1, -1$, thus A is diagonalizable.

(d) Yes. When $h = 2$, the 2-eigenspace is a plane because $A - 2I$ only has rank one. The 1-eigenspace has dimension at least one (thus exactly one), so the matrix A is diagonalizable.

19. One eigenvalue of the matrix $A = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}$ is $\lambda = 1 + 2i$.

(a) Which of the following is an eigenvector corresponding to the eigenvalue $\lambda = 1 + 2i$?

i. $\begin{pmatrix} -2i \\ 1 \end{pmatrix}$

ii. $\begin{pmatrix} 4 \\ 2 - 2i \end{pmatrix}$

iii. $\begin{pmatrix} 4 \\ 2i \end{pmatrix}$

iv. $\begin{pmatrix} -2i \\ 4 \end{pmatrix}$

v. $\begin{pmatrix} 4 \\ -2i \end{pmatrix}$

(b) What is the other eigenvalue of A?

i. $1 - 2i$

ii. $-1 + 2i$

iii. $-2 + i$

iv. $2 - i$

v. There is no other eigenvalue of A, because $\lambda = 1 + 2i$ has algebraic multiplicity 2.

vi. We need more information to determine what the second eigenvalue of A is.

Solution: Standard 5.5 example, similar to 5.5 Webwork #2 and practice exam #15.

(a) The answer is (v). The first row of $A - (1 + 2i)I$ is $(-2i \;\; -4)$, so one eigenvector is $\begin{pmatrix} 4 \\ -2i \end{pmatrix}$ by the 2×2 eigenvector trick.

(b) The answer is (i). The other eigenvalue of A is the complex conjugate of $1+2i$, namely $\lambda = 1-2i$.

20. Suppose A is a 3×3 matrix whose entries are real numbers, and suppose that $\lambda = 4-5i$ is an eigenvalue for A. How many real eigenvalues does A have?

(a) A has no real eigenvalues.

(b) A has exactly one real eigenvalue.

(c) A has exactly 2 real eigenvalues.

(d) Not enough information to tell how many real eigenvalues A has.

Solution: The answer is (b). Similar to #1a from the 5.4-5.5 worksheet and #4 from the 5.5 Webwork. We know that every 3×3 real matrix A must have at least one real eigenvalue (odd degree polynomial). From what is given we know that it already has $4 - 5i$ and (consequently) $4 + 5i$ as two eigenvalues, so A cannot have more than one real eigenvalue. Therefore, A has exactly one real eigenvalue.

A similar but slightly alternative way to see it: Applying the Fundamental Theorem of Algebra to the characteristic polynomial of A, we are guaranteed that A has exactly 3 eigenvalues counting multiplicities. We know $4 - 5i$ is an eigenvalue of A, therefore $4 + 5i$ is automatically an eigenvalue of A by section 5.5. This leaves us with just one eigenvalue remaining. Since non-real eigenvalues come in complex conjugate pairs (so we cannot have an additional non-real eigenvalue or repeat one of our non-real eigenvalues), our final eigenvalue must be real, so A has exactly one real eigenvalue.

One such matrix is $A = \begin{pmatrix} 11 & 2 & 0 \\ -37 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which has eigenvalues $\lambda = 4 \pm 5i$ and $\lambda = 1$.