Math 1553 Worksheet §§3.5-4.3

Solutions

1. True or false. Answer true if the statement is *always* true. Otherwise, answer false. If your answer is false, either give an example that shows it is false or (in the case of an incorrect formula) state the correct formula.

a) If A and B are $n \times n$ matrices and both are invertible, then the inverse of AB is $A^{-1}B^{-1}$.

b) If A is an $n \times n$ matrix and the equation $Ax = b$ has at least one solution for each b in $\mathbf{R}^n$, then the solution is *unique* for each b in $\mathbf{R}^n$.

c) If A is an $n \times n$ matrix and the equation $Ax = b$ has at most one solution for each b in $\mathbf{R}^n$, then the solution must be *unique* for each b in $\mathbf{R}^n$.

d) If A and B are invertible $n \times n$ matrices, then $A+B$ is invertible and $(A+B)^{-1} = A^{-1} + B^{-1}$.

e) If A is a 3×4 matrix and B is a 4×2 matrix, then the linear transformation Z defined by $Z(x) = ABx$ has domain $\mathbf{R}^3$ and codomain $\mathbf{R}^2$.

f) Suppose A is an $n \times n$ matrix and every vector in $\mathbf{R}^n$ can be written as a linear combination of the columns of A. Then A must be invertible.

g) If $\det(A) = 1$ and c is a scalar, then $\det(cA) = c\det(A)$.

Solution.

a) False. $(AB)^{-1} = B^{-1}A^{-1}$.

b) True. The first part says the transformation $T(x) = Ax$ is onto. Since A is $n \times n$, then it has n pivots. This is the same as saying A is invertible, and there is no free variable. Therefore, the equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^n$.

c) True. The first part says the transformation $T(x) = Ax$ is one-to-one (namely not multiple-to-one). Since A is $n \times n$, then it has n pivots. Then there is no free variable. Therefore, the equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^n$.

d) False. $A+B$ might not be invertible in the first place. For example, if $A = I_2$ and $B = -I_2$ then $A+B = 0$ which is not invertible. Even in the case when $A+B$ is invertible, it still might not be true that $(A+B)^{-1} = A^{-1} + B^{-1}$. For example, $(I_2 + I_2)^{-1} = (2I_2)^{-1} = \frac{1}{2}I_2$, whereas $(I_2)^{-1} + (I_2)^{-1} = I_2 + I_2 = 2I_2$.

e) False. In order for Bx to make sense, x must be in $\mathbf{R}^2$, and so Bx is in $\mathbf{R}^4$ and $A(Bx)$ is in $\mathbf{R}^3$. Therefore, the domain of Z is $\mathbf{R}^2$ and the codomain of Z is $\mathbf{R}^3$.

f) True. If the columns of A span $\mathbf{R}^n$, then A is invertible by the Invertible Matrix Theorem. We can also see this directly without quoting the IMT:

If the columns of A span $\mathbf{R}^n$, then A has n pivots, so A has a pivot in each row and column, hence its matrix transformation $T(x)=Ax$ is one-to-one and onto and thus invertible. Therefore, A is invertible.

g) False. By the properties of the determinant, scaling one row by c multiplies the determinant by c. When we take cA for an $n\times n$ matrix A, we are multiplying *each* row by c. This multiplies the determinant by c a total of n times. Thus, if A is $n\times n$ and $\det(A)=1$, then

$$\det(cA)=c^n\det(A)=c^n(1)=c^n.$$

2. Let $T:\mathbf{R}^2\to\mathbf{R}^2$ be rotation *clockwise* by 60°. Let $U:\mathbf{R}^2\to\mathbf{R}^2$ be the linear transformation satisfying $U(1,0)=(-2,1)$ and $U(0,1)=(1,0)$.

a) Find the standard matrix for the T and U, and compute the determinant of each matrix.

b) Find the standard matrix for the composition $U\circ T$ using matrix multiplication. Compute the determinant.

c) Find the standard matrix for the composition $T\circ U$ using matrix multiplication. Compute the determinant.

d) Is rotating clockwise by 60° and then performing U, the same as first performing U and then rotating clockwise by 60°?

e) What is the relation between the determinants of these matrices?

Solution.

To reduce confusion on notation, we are going to use T, U to denote standard matrices for linear transformation T, U.

a) The matrix for T is $\begin{pmatrix}\cos(-60^\circ) & -\sin(-60^\circ)\\ \sin(-60^\circ) & \cos(-60^\circ)\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$. Its determinant is $\frac{1}{2}*\frac{1}{2}-\frac{\sqrt{3}}{2}*(-\frac{\sqrt{3}}{2})=\frac{1}{4}+\frac{3}{4}=1$. (Alternatively we could use the fact that the determinant for a rotation matrix is always 1.)

The matrix for U is $\begin{pmatrix}U(e_1) & U(e_2)\end{pmatrix}=\begin{pmatrix}-2 & 1\\ 1 & 0\end{pmatrix}$. Its determinant is $-2*0-1*1=-1$.

b) The matrix for $U\circ T$ is

$$\begin{pmatrix}-2 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}=\begin{pmatrix}-1-\frac{\sqrt{3}}{2} & \frac{1}{2}-\sqrt{3}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{pmatrix}.$$

Its determinant is -1, as $\det(UT)=\det(U)\det(T)$

c) The matrix for $T\circ U$ is

$$\begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}-2 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}-1+\frac{\sqrt{3}}{2} & \frac{1}{2}\\ \frac{1}{2}+\sqrt{3} & -\frac{\sqrt{3}}{2}\end{pmatrix}.$$

Its determinant is −1 also, as $\det(TU) = \det(T)\det(U)$

d) No. In (a) and (b), we found that the standard matrices for $U \circ T$ and $T \circ U$ are different, so the transformations are different.

e) $\det(UT)$ and $\det(TU)$ are the same, since the determinant of the product of two matrices *is* commutative, unlike the product itself. Specifically, $\det(UT) = \det(TU) = \det(T) \times \det(U)$.

3. Let $A = \begin{pmatrix} 7 & 1 & 4 & 1 \\ -1 & 0 & 0 & 6 \\ 9 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

a) Compute $\det(A)$.

b) Compute $\det(A^{-1})$ without doing any more work.

c) Compute $\det((A^T)^5)$ without doing any more work.

d) Find the volume of the parallelepiped formed by the columns of A.

Solution.

a) The second column has three zeros, so we expand by cofactors:

$$\det\begin{pmatrix} 7 & 1 & 4 & 1 \\ -1 & 0 & 0 & 6 \\ 9 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\det\begin{pmatrix} -1 & 0 & 6 \\ 9 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$$

Now we expand the second column by cofactors again:

$$\cdots = -2\det\begin{pmatrix} -1 & 6 \\ 0 & -1 \end{pmatrix} = (-2)(-1)(-1) = -2.$$

b) From our notes, we know $\det(A^{-1}) = \dfrac{1}{\det(A)} = -\dfrac{1}{2}$.

c) $\det(A^T) = \det(A) = -2$. By the multiplicative property of determinants, if B is any $n \times n$ matrix, then $\det(B^n) = (\det B)^n$, so

$$\det((A^T)^5) = (\det A^T)^5 = (-2)^5 = -32.$$

d) Volume of the parallelepiped is $|\det(A)| = 2$