Section 1.1

Solving systems of equations

Outline of Section 1.1

- Learn what it means to solve a system of linear equations
- Describe the solutions as points in $\mathbb{R}^n$
- Learn what it means for a system of linear equations to be inconsistent

Solving equations

Solving equations

What does it mean to solve an equation?

$2x = 10$

$x + y = 1$

$x + y + z = 0$

Find one solution to each. Can you find all of them?

A solution is a *list* of numbers. For example $(3, -4, 1)$.

Solving equations

What does it mean to solve a system of equations?

$$\begin{aligned} x + y &= 2 \\ y &= 1 \end{aligned}$$

What about...

$$\begin{aligned} x + y + z &= 3 \\ x + y - z &= 1 \\ x - y + z &= 1 \end{aligned}$$

Is $(1, 1, 1)$ a solution? Is $(2, 0, 1)$ a solution? What are all the solutions?

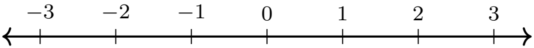
Soon, you will be able to see just by looking that there is exactly one solution.

$$\mathbb{R}^n$$

$\mathbb{R}^n$

$\mathbb{R}$: denotes the set of all real numbers

Geometrically, this is the *number line*.

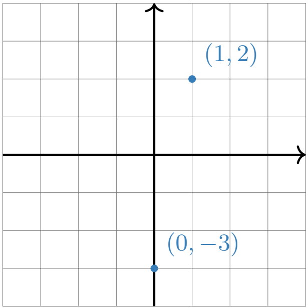


$$\mathbb{R}^n = \text{all ordered } n\text{-tuples (or lists) of real numbers } (x_1, x_2, x_3, \ldots, x_n)$$

Solutions to systems of equations are exactly points in $\mathbb{R}^n$. In other words, $\mathbb{R}^n$ is where our solutions will live (the n depends on the system of equations).

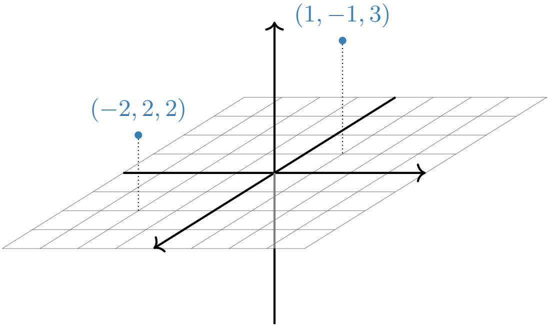
$\mathbb{R}^n$

When $n = 2$, we can visualize of $\mathbb{R}^2$ as the *plane*.



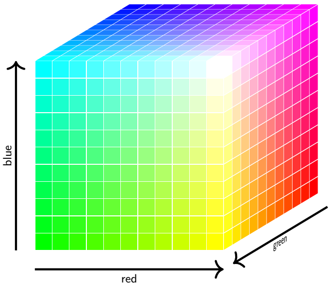
$\mathbb{R}^n$

When $n = 3$, we can visualize $\mathbb{R}^3$ as the *space* we (appear to) live in. This is because every point in space can be represented by an ordered triple of real numbers, namely, its x-, y-, and z-coordinates.



$\mathbb{R}^n$

We can think of the space of all *colors* as (a subset of) $\mathbb{R}^3$: All colors you can see can be described by three quantities: the amount of red, green, and blue light in that color. Therefore, we can use points in $\mathbb{R}^3$ to *label* all colors: the point $(.2, .4, .9)$ labels the color with 20% red, 40% green, and 90% blue.



$\mathbb{R}^n$

So what is $\mathbb{R}^4$? or $\mathbb{R}^5$? or $\mathbb{R}^n$?

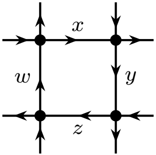
...go back to the *definition*: ordered n-tuples of real numbers

$$(x_1, x_2, x_3, \ldots, x_n).$$

They're still "geometric" spaces, in the sense that our intuition for $\mathbb{R}^2$ and $\mathbb{R}^3$ sometimes extends to $\mathbb{R}^n$, but they're harder to visualize.

Last time we could have used $\mathbb{R}^3$ to describe a rabbit population in a given year: (first year, second year, third year).

Similarly, we could have used $\mathbb{R}^4$ to label the amount of traffic (x, y, z, w) passing through four streets.



We'll make definitions and state theorems that apply to any $\mathbb{R}^n$, but we'll only draw pictures in $\mathbb{R}^2$ and $\mathbb{R}^3$.

$\mathbb{R}^n$ and QR codes

This is a 21×21 QR code. We can also think of this as an element of $\mathbb{R}^n$.

How? Which n?

What about a greyscale image?

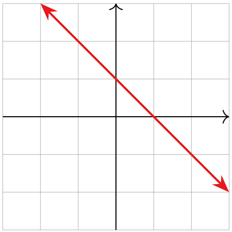
This is a powerful idea: instead of thinking of a QR code as 441 pieces of information, we think of it as one piece of information.

Visualizing solutions: a preview

One Linear Equation

What does the solution set of a linear equation look like?

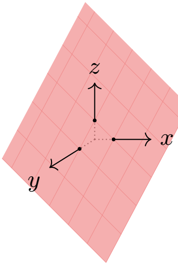
$x + y = 1 \rightsquigarrow$ a line in the plane: $y = 1 - x$



One Linear Equation

What does the solution set of a linear equation look like?

$x + y + z = 1 \rightsquigarrow$ a plane in space:



One Linear Equation
Continued

What does the solution set of a linear equation look like?

$x + y + z + w = 1 \rightsquigarrow$ a "3-plane" in "4-space"...

Systems of Linear Equations

What does the solution set of a *system* of more than one linear equation look like? For example:

$$x - 3y = -3$$
$$2x + y = 8$$

. . . is the *intersection* of two lines, which is a *point* in this case.

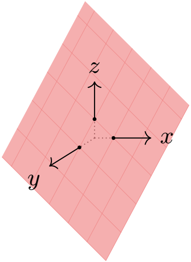
Q: What are the other possibilities for two equations with two variables?
We covered this on our first worksheet. The three possibilities are:
1. No solutions (two different parallel lines).
2. Exactly one solution (two non-parallel lines).
3. Infinitely many solutions (two identical lines).

Q: What if there are more variables? More equations?
This is a fundamental question we will study throughout chapter 1. The techniques of section 1.2 will be crucial.

Poll

Is the plane in $\mathbb{R}^3$ from the previous example equal to $\mathbb{R}^2$? Is the xy-plane in $\mathbb{R}^3$ equal to $\mathbb{R}^2$?

1. yes + yes
2. yes + no
3. no + yes
4. no + no



Answer. No + no. To describe any point in $\mathbb{R}^3$, you need three numbers, not a list of two numbers. The xy-plane in $\mathbb{R}^3$ is the set of all triples $(x, y, 0)$. So it's not $\mathbb{R}^2$ even though it is *like* $\mathbb{R}^2$.

Consistent versus Inconsistent

We say that a system of linear equations is consistent if it has a solution and inconsistent otherwise.

$$x + y = 1$$
$$x + y = 2$$

Why is this inconsistent?
Algebraically, we get $1 = 2$, which is impossible. Geometrically, we see the lines $y = 1 - x$ and $y = 2 - x$ are different parallel lines, so they never intersect.

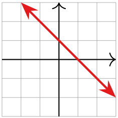
What are other examples of inconsistent systems of linear equations?
Similar to above, we could take two different parallel planes in $\mathbb{R}^3$:

$$x - y + z = 1$$

$$x - y + z = 2.$$

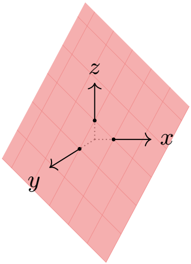
Parametric form

The equation $x + y = 1$ is an implicit equation for the line in the picture.



It also has a parametric form: $(x, 1 - x)$

Similarly the equation $x + y + z = 1$ is an implicit equation. One parametric form is: $(x, y, 1 - x - y)$.



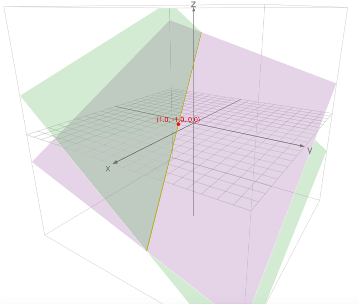
What is an implicit equation and a parametric form for the xy-plane in $\mathbb{R}^3$?

Parametric form

The system of equations

$$\begin{aligned} 2x + y + 12z &= 1 \\ x + 2y + 9z &= -1 \end{aligned}$$

is the implicit form for the line of intersection in the picture.



The line of intersection also has a parametric form: $(1 - 5z, -1 - 2z, z)$

We think of the former as being the problem and the latter as being the explicit solution. One of our first tasks this semester is to learn how to go from the implicit form to the parametric form.

Augmented Matrices

We can express systems of equations using shorthand notation, by using augmented matrices.

Instead of writing the linear equations in full, we record the coefficients, in order, in the left side of the matrix. The bar separates the left and right side of the equations like an equals sign.

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned} \quad \overset{\text{becomes}}{\rightsquigarrow} \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Summary of Section 1.1

- A solution to a system of linear equations in n variables is a point in $\mathbb{R}^n$.
- The set of all solutions to a single equation in n variables is an $(n-1)$-dimensional plane in $\mathbb{R}^n$
- The set of solutions to a system of m linear equations in n variables is the intersection of m of these $(n-1)$-dimensional planes in $\mathbb{R}^n$.
- A system of equations with no solutions is said to be inconsistent.
- Line and planes have implicit equations and parametric forms.
- We can use augmented matrices to save time and effort writing systems of linear equations.