Section 1.2

Row Reduction

Review from 1.1: Solving Systems of Equations

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

This is the kind of problem we'll talk about for the first half of the course.

- A **solution** is a list of numbers $x, y, z, \ldots$ that makes *all* of the equations true.
- The **solution set** is the collection of all solutions.
- **Solving** the system means finding the solution set in a "parameterized" form.

Definition

A system of equations is called **inconsistent** if it has no solution. It is **consistent** otherwise.

What is a *systematic* way to solve a system of equations?

Review from 1.1: Solving Systems of Equations

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

What strategies do you know?

- Substitution
- Elimination

Both are perfectly valid, but only elimination scales well to large numbers of equations.

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Elimination method: in what ways can you manipulate the equations?

- Multiply an equation by a nonzero number. **(scale)**
- Add a multiple of one equation to another. **(replacement)**
- Swap two equations. **(swap)**

Solving Systems of Equations

Better notation

It sure is a pain to have to write x, y, z, and $=$ over and over again.

Matrix notation: write just the numbers, in a box, instead!

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 6 \\ 2x & - & 3y & + & 2z & = & 14 \\ 3x & + & y & - & z & = & -2 \end{array} \xrightarrow{\text{becomes}} \left(\begin{array}{rrr|r} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

This is called an **(augmented) matrix**. Our equation manipulations become **elementary row operations**:

- Multiply all entries in a row by a nonzero number. **(scale)**
- Add a multiple of each entry of one row to the corresponding entry in another. **(row replacement)**
- Swap two rows. **(swap)**

Row Equivalence

Important

The process of doing row operations to a matrix does not change the solution set of the corresponding linear equations!

Definition

Two matrices are called **row equivalent** if one can be obtained from the other by doing some number of elementary row operations.

So the linear equations of row-equivalent matrices have the *same solution set*.

Row Operations, Fundamental Example

Example

Solve the system of equations

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned}$$

Start:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

Goal: we want our elimination method to eventually produce a system of equations like

$$\begin{aligned} x \qquad\qquad &= A \\ y \qquad &= B \\ z &= C \end{aligned} \qquad \text{or in matrix form,} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array}\right)$$

So we need to do row operations that make the start matrix look like the end one.

Strategy (preliminary): fiddle with it so we only have ones and zeros. [animated]

Row Operations
Continued

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right)$$

We want these to be zero.
So we subract multiples of the first row.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right) \xrightarrow{R_2 \longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 \div -5} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -7 & -4 & 2 \end{array}\right) \xrightarrow{R_3 = R_3 + 7R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right)$$

We want these to be zero.

It would be nice if this were a 1.
We could divide by -7, but that would produce ugly fractions.

Let's swap the last two rows first.

Row Operations

Continued

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 10 & 30 \end{array}\right) \xrightarrow{R_3 = R_3 \div 10} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

We want these to be zero.

Let's make this a 1 first.

$$\xrightarrow{R_2 = R_2 - 2R_3,\ R_1 = R_1 - 3R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{\text{translates into}} \begin{aligned} x &= 1 \\ y &= -2 \\ z &= 3 \end{aligned}$$

Success!

Check:

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x - 3y + 2z &= 14 \\ 3x + y - z &= -2 \end{aligned} \xrightarrow{\text{substitute solution}} \begin{aligned} 1 + 2 \cdot (-2) + 3 \cdot 3 &= 6 \\ 2 \cdot 1 - 3 \cdot (-2) + 2 \cdot 3 &= 14 \\ 3 \cdot 1 + (-2) - 3 &= -2 \end{aligned} \checkmark$$

Row Echelon Form

Let's come up with an *algorithm* for turning an arbitrary matrix into a "solved" matrix. What do we mean by "solved"?

A matrix is in **row echelon form** if

1. All zero rows are at the bottom.
2. Each leading nonzero entry of a row is to the *right* of the leading entry of the row above.
3. Below a leading entry of a row, all entries are *zero*.

Picture:

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{aligned} \star &= \text{any number} \\ \boxed{\star} &= \text{any nonzero number} \end{aligned}$$

Definition

A **pivot** $\boxed{\star}$ is the first nonzero entry of a row of a matrix. A **pivot column** is a column containing a pivot of a matrix *in row echelon form*.

Reduced Row Echelon Form

A matrix is in **reduced row echelon form** if it is in row echelon form, and in addition,

4. The pivot in each nonzero row is equal to 1.
5. Each pivot is the only nonzero entry in its column.

Picture:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\star =$ any number

$1 =$ pivot

Note: Echelon forms do not care whether or not a column is augmented. Just ignore the vertical line.

Question

Can every matrix be put into reduced row echelon form only using row operations?

Answer: Yes!

Why is this the "solved" version of the matrix from the fundamental example?

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

It translates into

$$\begin{aligned} x &= 1 \\ y &= -2 \\ z &= 3, \end{aligned}$$

which is clearly the solution.

But what happens if there are fewer pivots than rows?

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

... parametrized solution set (later).

An Inconsistent Example

Example

Solve the system of equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned}$$

Let's try doing row operations: [interactive row reducer]

First clear these by subtracting multiples of the first row.

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{array}\right) \xrightarrow{R_2 = R_2 - 3R_1} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 4 & 5 & 9 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - 4R_1} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array}\right)$$

Now clear this by subtracting the second row.

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow{R_3 = R_3 - R_2} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right)$$

An Inconsistent Example

Continued

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right) \xrightarrow{\text{translates into}} \begin{aligned} x + y &= 2 \\ y &= -1 \\ {\color{red}0} &{\color{red}= 2} \end{aligned}$$

In other words, the original equations

$$\begin{aligned} x + y &= 2 \\ 3x + 4y &= 5 \\ 4x + 5y &= 9 \end{aligned} \qquad \text{have the same solutions as} \qquad \begin{aligned} x + y &= 2 \\ y &= -1 \\ 0 &= 2 \end{aligned}$$

But the latter system obviously has no solutions (there is no way to make them all true), so our original system has no solutions either.

In terms of the augmented matrix, what went wrong is that we had a pivot in the rightmost column. This means the side left of the equals sign is 0, but the right side is nonzero, which is impossible.

Poll

Poll

Which of the following matrices are in reduced row echelon form?

A. $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ B. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

C. $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ D. $\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$ E. $\begin{pmatrix} 0 & 1 & 8 & 0 \end{pmatrix}$

F. $\left(\begin{array}{cc|c} 1 & 17 & 0 \\ 0 & 0 & 1 \end{array}\right)$

Answer: B, D, E, F.

Note that A is in row echelon form though.

Extra example 1

Translate the equation to an augmented matrix and put the matrix in RREF. Label all pivots. Feel free to use the ▸ Interactive Row Reducer.

$$x_1 + 2x_2 + 2x_3 - x_4 = 4$$

$$2x_1 + 4x_2 + x_3 - 2x_4 = -1$$

$$-x_1 - 2x_2 - x_3 + x_4 = -1$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 4 \\ 2 & 4 & 1 & -2 & -1 \\ -1 & -2 & -1 & 1 & -1 \end{array}\right) \xrightarrow[R_3 = R_3 + R_1]{R_2 = R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 4 \\ 0 & 0 & -3 & 0 & -9 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & -3 & 0 & -9 \end{array}\right)$$

$$\xrightarrow[R_1 = R_1 - 2R_2]{R_3 = R_3 + 3R_2} \left(\begin{array}{cccc|c} \boxed{1} & 2 & 0 & -1 & -2 \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Extra example 2

Translate the equation to an augmented matrix and put the matrix in RREF. Label all pivots. Feel free to use the ▸ Interactive Row Reducer.

$$x_3 + 3x_4 = 7$$

$$2x_1 - 6x_3 - 6x_4 = -6$$

$$4x_1 - 9x_3 - 3x_4 + x_5 = 8.$$

$$\left(\begin{array}{ccccc|c} 0 & 0 & 1 & 3 & 0 & 7 \\ 2 & 0 & -6 & -6 & 0 & -6 \\ 4 & 0 & -9 & -3 & 1 & 8 \end{array}\right) \xrightarrow{\text{work}} \left(\begin{array}{ccccc|c} \boxed{1} & 0 & 0 & 6 & 0 & 18 \\ 0 & 0 & \boxed{1} & 3 & 0 & 7 \\ 0 & 0 & 0 & 0 & \boxed{1} & -1 \end{array}\right)$$

Summary

- We can more easily do elimination with matrices. The allowable moves are row swaps, row scales, and row replacements. This is called row reduction.
- A matrix in row echelon form corresponds to a system of linear equations that we can easily solve by back substitution.
- A matrix in reduced row echelon form corresponds to a system of linear equations that we can easily solve just by looking.
- We have an algorithm for row reducing a matrix to reduced row echelon form.
- The reduced row echelon form of a matrix is unique.
- Two matrices that differ by row operations are called row equivalent. Row-equivalent systems have the *same solution set*.
- A system of equations is inconsistent exactly when the corresponding augmented matrix has a pivot in the last column.