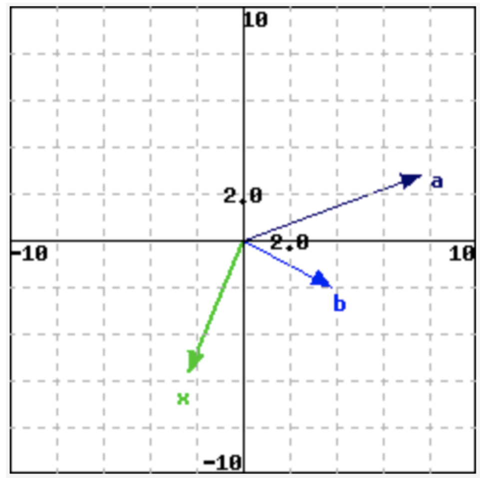
Supplemental problems: §2.1, §2.2, §2.3

1. Consider the following vectors x, a, and b.



Can we write x as a linear combination of a and b?

a) Formulate this question as a vector equation.

b) Formulate this question as a matrix equation.

c) Give an approximate answer to the question using geometric intuition rather than attempting to solve a system of equations.

Solution.

a) $c_1 a + c_2 b = x$.

b) $\begin{pmatrix} | & | \\ a & b \\ | & | \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = x.$

c) If we stretch b by a little bit, then subtract a, we will get x. This is roughly

$$-a + \frac{3}{2}b = x.$$

(in reality, $x = -a + 1.4b$).

2. Consider the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 1 & -3 & -4 & -9 \\ 3 & -1 & 8 & 9 \end{array}\right)$$

Question: Does the corresponding linear system have a solution? If so, what is the solution set?

a) Formulate this question as a vector equation.

b) Formulate this question as a system of linear equations.

c) Formulate this question as a matrix equation.

d) What does this mean in terms of spans?

e) Answer the question using the interactive demo.

f) Answer the question using row reduction.

g) Find a **different** solution in parts (e) and (d).

Solution.

a) What are the solutions to the following vector equation?

$$x\begin{pmatrix}2\\1\\3\end{pmatrix}+y\begin{pmatrix}-2\\-3\\-1\end{pmatrix}+z\begin{pmatrix}2\\-4\\8\end{pmatrix}=\begin{pmatrix}0\\-9\\9\end{pmatrix}$$

b) What is the solution set of the following linear system?

$$\begin{aligned}2x-2y+2z&=0\\x-3y-4z&=-9\\3x-y+8z&=9\end{aligned}$$

c) There exists a solution if and only if $\begin{pmatrix}0\\-9\\9\end{pmatrix}$ is in Span $\left\{\begin{pmatrix}2\\1\\3\end{pmatrix},\begin{pmatrix}-2\\-3\\-1\end{pmatrix},\begin{pmatrix}2\\-4\\8\end{pmatrix}\right\}$.

d) As a matrix equation:

$$\begin{pmatrix}2&-2&2\\1&-3&-4\\3&-1&8\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\-9\\9\end{pmatrix}.$$

f) Row reducing yields

$$\left(\begin{array}{ccc|c}1&0&7/2&9/2\\0&1&5/2&9/2\\0&0&0&0\end{array}\right).$$

Hence z is a free variable, so the solution in parametric form is

$$\begin{aligned}x&=\tfrac{9}{2}-\tfrac{7}{2}z\\y&=\tfrac{9}{2}-\tfrac{5}{2}z.\end{aligned}$$

Taking $z=0$ yields the solution $x=y=9/2$.

g) Taking $z=1$ yields the solution $x=1, y=2$.

3. Let $v_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$ $\quad w = \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix}$.

Question: Is w a linear combination of v_1 and v_2? In other words, is w in $\mathrm{Span}\{v_1, v_2\}$?

a) Formulate this question as a vector equation.

b) Formulate this question as a system of linear equations.

c) Formulate this question as an augmented matrix.

d) Formulate this question as a matrix equation.

e) Answer the question using the interactive demo.

f) Answer the question using row reduction.

Solution.

a) Does the following vector equation have a solution?

$$x \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + y \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix}$$

b) Does the following linear system have a solution?

$$\begin{aligned} 2x - 2y &= 2 \\ x - 3y &= -4 \\ 3x - y &= 8 \end{aligned}$$

c) As an augmented matrix:

$$\left(\begin{array}{cc|c} 2 & -2 & 2 \\ 1 & -3 & -4 \\ 3 & -1 & 8 \end{array}\right)$$

d) As a matrix equation:

$$\begin{pmatrix} 2 & -2 \\ 1 & -3 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 8 \end{pmatrix}.$$

f) Row reducing yields

$$\left(\begin{array}{cc|c} 1 & 0 & 7/2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{array}\right)$$

so $x = 7/2$ and $y = 5/2$.

4. Let

$$A = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$$

Is b in the span of the columns of A? In other words, is b a linear combination of the columns of A? Justify your answer.

Solution.

Let v_1, v_2, and v_3 be the columns of A. We are asked to determine whether there are scalars x_1, x_2, and x_3 so that $x_1v_1 + x_2v_2 + x_3v_3 = b$, which means

$$\begin{aligned} x_1 \qquad\quad + 5x_3 &= 2 \\ -2x_1 + x_2 - 6x_3 &= -1 \\ 2x_2 + 8x_3 &= 6 \end{aligned}$$

We translate the system of linear equations into an augmented matrix, and row reduce it:

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The right column is not a pivot column, so the system is consistent. Therefore, b is in the span of the columns of A (in other words, b is a linear combination of the columns of A).

We weren't asked to solve the equation explicitly, but if we wanted to do so, we would use the RREF of the matrix above to write

$$x_1 = 2 - 5x_3 \qquad x_2 = 3 - 4x_3 \qquad x_3 = x_3 \ \ (x_3 \text{ is free}).$$

In fact, we can take $x_1 = 2, x_2 = 3$, and $x_3 = 0$, to write

$$b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}.$$

5. Consider the vector equation

$$x\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + y\begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} + z\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix}.$$

Question: Is there a solution? If so, what is the solution set?

a) Formulate this question as an augmented matrix.

b) Formulate this question as a system of linear equations.

c) Formulate this question as a matrix equation.

d) What does this mean in terms of spans?

e) Answer the question using the interactive demo.

f) Answer the question using row reduction.

Solution.

a) As an augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & -2 & 3 & -5 \\ 1 & -1 & 0 & -1 \\ 3 & -1 & 4 & -2 \end{array}\right)$$

b) What is the solution set of the following linear system?

$$\begin{aligned} 2x - 2y + 3z &= -5 \\ x - y &= -1 \\ 3x - y + 4z &= -2 \end{aligned}$$

c) There exists a solution if and only if $\begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix}$ is in Span $\left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \right\}$.

d) As a matrix equation:

$$\begin{pmatrix} 2 & -2 & 3 \\ 1 & -1 & 0 \\ 3 & -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ -2 \end{pmatrix}.$$

f) Row reducing yields

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & -1 \end{array}\right),$$

so $x = 3/2$, $y = 5/2$, and $z = -1$.

6. Let $v_1 = \begin{pmatrix} 1 \\ k \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, and $b = \begin{pmatrix} 1 \\ h \end{pmatrix}$.

a) Find all values of h and k so that $x_1 v_1 + x_2 v_2 = b$ has infinitely many solutions.

b) Find all values of h and k so that b is *not* in Span$\{v_1, v_2\}$.

c) Find all values of h and k so that there is exactly one way to express b as a linear combination of v_1 and v_2.

Solution.

Each part uses the row-reduction

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ k & 4 & h \end{array}\right) \xrightarrow{R_2 = R_2 - kR_1} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 4+k & h-k \end{array}\right).$$

a) The system $\left(\begin{array}{cc|c} v_1 & v_2 & b \end{array}\right)$ has infinitely many solutions if and only if the right column is not a pivot column and there is at least one free variable. This means that $4 + k = 0$ and $h - k = 0$, so $k = -4$ and $h = k$, thus $\boxed{k = -4 \text{ and } h = -4}$.

b) The right column is a pivot column when $4+k=0$ and $h-k\neq 0$. Thus $\boxed{k=-4 \text{ and } h\neq -4}$.

c) The system will have a unique solution when the right column is not a pivot column but both other columns are pivot columns. This is when $4+k\neq 0$, so $\boxed{k\neq -4 \text{ and } h \text{ is any real number}}$.

7. Decide if each of the following statements is true or false. If it is true, prove it; if it is false, provide a counterexample.

a) Every set of four or more vectors in $\mathbf{R}^3$ will span $\mathbf{R}^3$.

b) The span of any set contains the zero vector.

Solution.

a) This is **false.** For instance, the vectors

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}3\\0\\0\end{pmatrix}, \begin{pmatrix}4\\0\\0\end{pmatrix}\right\}$$

only span the x-axis.

b) This is **true.** We have

$$0 = 0\cdot v_1 + 0\cdot v_2 + \cdots + 0\cdot v_p.$$

Aside: the span of the empty set is equal to $\{0\}$, because 0 is the empty sum, i.e. the sum with no summands. Indeed, if you add the empty sum to a vector v, you get $v+$(no other summands), which is just v; and the only vector which gives you v when you add it to v, is 0. (If you find this argument intriguing, you might want to consider taking abstract math courses later on.)

8. Is $\begin{pmatrix}0\\1\\0\end{pmatrix}$ in the span of $\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\3\\1\end{pmatrix}$? Justify your answer.

Solution.

No. We row-reduce the corresponding augmented matrix to get

$$\left(\begin{array}{cc|c}0 & 2 & 0\\ 1 & 3 & 1\\ 1 & 1 & 0\end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right)$$

which is inconsistent since it has a pivot in the right column.

9. Acme Widgets, Gizmos, and Doodads has two factories. Factory A makes 10 widgets, 3 gizmos, and 2 doodads every hour, and factory B makes 4 widgets, 1 gizmo, and 1 doodad every hour.

a) If factory A runs for a hours and factory B runs for b hours, how many widgets, gizmos, and doodads are produced? Express your answer as a vector equation.

b) A customer places an order for 16 widgets, 5 gizmos, and 3 doodads. Can Acme fill the order with no widgets, gizmos, or doodads left over? If so, how many hours do the factories run? If not, why not?

Solution.

a) Let w, g, and d be the number of widgets, gizmos, and doodads produced.

$$\begin{pmatrix} w \\ g \\ d \end{pmatrix} = a \begin{pmatrix} 10 \\ 3 \\ 2 \end{pmatrix} + b \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}.$$

b) We need to solve the vector equation

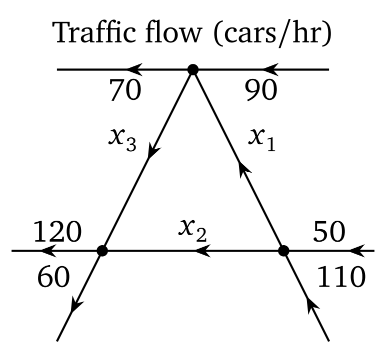
$$\begin{pmatrix} 16 \\ 5 \\ 3 \end{pmatrix} = a \begin{pmatrix} 10 \\ 3 \\ 2 \end{pmatrix} + b \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}.$$

We put it into an augmented matrix and row reduce:

$$\left(\begin{array}{cc|c} 10 & 4 & 16 \\ 3 & 1 & 5 \\ 2 & 1 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 3 & 1 & 5 \\ 2 & 1 & 3 \\ 10 & 4 & 16 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 10 & 4 & 16 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 10 & 4 & 16 \end{array}\right)$$
$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right).$$

These equations are consistent, but they tell us that factory B would have to run for -1 hours! Therefore it can't be done.

10. The diagram below represents traffic in a city.



a) Write a system of three linear equations whose solution would give the values of x_1, x_2, and x_3. Do not solve it.

b) Write the system of equations as a vector equation. Do not solve it.

Solution.

a) The number of cars leaving an intersection must equal the number of cars entering.

$$x_3 + 70 = x_1 + 90$$
$$x_1 + x_2 = 160$$
$$x_2 + x_3 = 180.$$

Or:

$$\begin{aligned} -x_1 \phantom{{}+x_2} + x_3 &= 20 \\ x_1 + x_2 \phantom{{}+x_3} &= 160 \\ x_2 + x_3 &= 180. \end{aligned}$$

b) $x_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 160 \\ 180 \end{pmatrix}.$

11. True or false. If the statement is *ever* false, answer false. Justify your answer.

a) A matrix equation $Ax = b$ is consistent if A has a pivot in every column.

b) If an $m \times n$ matrix A has fewer than n pivots and b is in $\mathbf{R}^m$, then $Ax = b$ has infinitely many solutions.

c) Suppose A is a 3×3 matrix and there is a vector y in $\mathbf{R}^3$ so that $Ax = y$ does not have a solution. Is it possible that there is a z in $\mathbf{R}^3$ so that the equation $Ax = z$ has a *unique* solution? Justify your answer.

d) There is a matrix A and a nonzero vector b so that the solution set of $Ax = b$ is a plane through the origin.

e) Suppose A is an $m \times n$ matrix and b is in $\mathbf{R}^m$. If the columns of A span $\mathbf{R}^m$, then $Ax = b$ must be consistent.

Solution.

a) False. For example, the system $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ has no solution, even though the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has a pivot in every column. However, the system is guaranteed to be consistent if A has a pivot in every **row**.

b) False: For example, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ has an A with one pivot but has no solutions.

c) False. Since $Ax = y$ is inconsistent for some y in $\mathbf{R}^3$, it follows that A has at least one row without a pivot, so A has at most 2 pivots. Therefore, at least one of the three columns of A will not have a pivot, so if an equation $Ax = z$

is consistent, the system will have a free variable and thus infinitely many solutions.

d) False. If the solution set to $Ax = b$ is a plane through the origin, then $x = 0$ is a solution, so $b = A(0)$ and therefore $b = 0$.

e) True. The span of the columns of A is exactly the set of all v for which $Ax = v$ is consistent. Since the span is $\mathbf{R}^m$, the matrix equation is consistent no matter what b is.