Supplemental problems: §3.1

1. Review from 2.6-2.9. Fill in the blanks: If A is a 7×6 matrix and the solution set for $Ax = 0$ is a plane, then the column space of A is a ________-dimensional subspace of $\mathbf{R}^{\square}$.

2. Review from 2.6-2.9: Consider the matrix A below and its RREF:

$$A = \begin{pmatrix} 1 & 2 & -1 & -1 \\ -2 & -4 & -6 & 2 \\ 1 & 2 & -5 & -1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Write a basis for Col A.

b) Find a basis for Nul A.

c) Is there a matrix B so that $\mathrm{Col}(B) = \mathrm{Nul}(A)$? If yes, write such a B. If not, justify why no such matrix B exists.

3. Suppose T is a matrix transformation and the range of T is the subspace

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x - 3y + 4z = 0 \right\}$$

of $\mathbf{R}^3$, which contains the vectors $v_1 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$. Is $\{v_1, v_2\}$ a basis for the range of T?

4. True or false. If the statement is *always* true, answer TRUE. Otherwise, circle FALSE.

a) The matrix transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ performs reflection across the x-axis in $\mathbf{R}^2$. TRUE FALSE

b) The matrix transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ performs rotation counter-clockwise by 90° in $\mathbf{R}^2$. TRUE FALSE

5. Let T be the matrix transformation $T(x) = Ax$, where $A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ -1 & 0 & -1 & -2 \\ 2 & 2 & 4 & 2 \end{pmatrix}$.
What is the domain of T? What is its codomain? Find a basis for the range of T and a basis for the kernel of T (the kernel of T is the set of all vectors satisfying $T(x) = 0$).

6. The matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ has RREF $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$. Define a matrix transformation by $T(x) = Ax$. Is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ a basis for the range of T?

7. In each case, a matrix is given below.

Match each matrix to its corresponding transformation (choosing from (i) through (viii)) by writing that roman numeral next to the matrix. Note there are four matrices and eight options, so not every option is used.

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

(i) Reflection across x-axis
(ii) Reflection across y-axis
(iii) Scaling by a factor of 2
(iv) Scaling by a factor of 1/2
(v) Rotation counterclockwise by $\pi/2$ radians
(vi) Rotation clockwise by $\pi/2$ radians
(vii) Projection onto the x-axis
(viii) Projection onto the y-axis

Supplemental problems: §3.2

1. Let A be a 3×4 matrix with column vectors v_1, v_2, v_3, v_4, and suppose $v_2 = 2v_1 - 3v_4$. Consider the matrix transformation $T(x) = Ax$.

a) Is it possible that T is one-to-one? If yes, justify why. If no, find distinct vectors v and w so that $T(v) = T(w)$.

b) Is it possible that T is onto? Justify your answer.

2. **a)** Which of the following are onto transformations? (Check all that apply.)

☐ $T : \mathbf{R}^3 \to \mathbf{R}^3$, reflection over the xy-plane

☐ $T : \mathbf{R}^3 \to \mathbf{R}^3$, projection onto the xy-plane

☐ $T : \mathbf{R}^3 \to \mathbf{R}^2$, project onto the xy-plane, forget the z-coordinate

☐ $T : \mathbf{R}^2 \to \mathbf{R}^2$, scale the x-direction by 2

b) Let A be a square matrix and let $T(x) = Ax$. Which of the following guarantee that T is onto? (Check all that apply.)

☐ T is one-to-one

☐ $Ax = 0$ is consistent

3. Find all real numbers h so that the transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ given by

$$T(v) = \begin{pmatrix} -1 & 0 & 2-h \\ h & 0 & 3 \end{pmatrix} v$$

is onto.

4. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a matrix transformation $T(x) = Ax$. Which of the following conditions guarantee that T *must* be one-to-one? Circle all that apply.
(i) A has m pivots.
(ii) The columns of A are linearly independent.
(iii) For each input vector x in $\mathbf{R}^n$, there is exactly one output vector $T(x)$ in $\mathbf{R}^m$.
(iv) The equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^m$.

5. Answer each question.

a) Suppose $S : \mathbf{R}^3 \to \mathbf{R}^2$ is the matrix transformation $S(x) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} x$.

Is S one-to-one? YES NO

Is S onto? YES NO

b) Suppose $T : \mathbf{R}^2 \to \mathbf{R}^2$ is given by $T(x, y) = (x - y,\ x - y)$.
Is T one-to-one? YES NO

Is T onto? YES NO

c) Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ is a one-to-one matrix transformation. Which one of the following *must* be true? (cicle one)

$m = n$ $\quad m < n$ $\quad m \leq n$ $\quad m > n$ $\quad m \geq n$

6. Which of the following transformations are onto?
Circle all that apply.

a) $T : \mathbf{R}^2 \to \mathbf{R}^2$ that rotates counterclockwise by $\dfrac{\pi}{12}$ radians.

b) The transformation $T(x) = Ax$, where A is a 4×3 matrix with three pivots.

c) $T : \mathbf{R}^3 \to \mathbf{R}^3$ that reflects across the yz-plane.