Supplemental problems: Chapter 4, Determinants

1. If A is an $n \times n$ matrix, is it necessarily true that $\det(-A) = -\det(A)$? Justify your answer.

Solution.

No. Since $\det(cA) = c^n \det(A)$, we see $\det(-A) = (-1)^n \det(A)$, so $\det(-A) = \det(A)$ if n is even and $\det(-A) = -\det(A)$ if n is odd.

2. Let A be an $n \times n$ matrix.

a) Using cofactor expansion, explain why $\det(A) = 0$ if A has a row or a column of zeros.

b) Using cofactor expansion, explain why $\det(A) = 0$ if A has adjacent identical columns.

Solution.

a) If A has zeros for all entries in row i (so $a_{i1} = a_{i2} = \cdots = a_{in} = 0$), then the cofactor expansion along row i is

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = 0 \cdot C_{i1} + 0 \cdot C_{i2} + \cdots + 0 \cdot C_{in} = 0.$$

Similarly, if A has zeros for all entries in column j, then the cofactor expansion along column j is the sum of a bunch of zeros and is thus 0.

b) If A has identical adjacent columns, then the cofactor expansions will be identical except that one expansion's terms for $\det(A)$ will have plus signs where the other expansion's terms for $\det(A)$ have minus signs (due to the $(-1)^{\text{power}}$ factors) and vice versa.

Therefore, $\det(A) = -\det(A)$, so $\det A = 0$.

3. Find the volume of the parallelepiped in $\mathbf{R}^4$ naturally determined by the vectors

$$\begin{pmatrix} 4 \\ 1 \\ 3 \\ 8 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 7 \\ 0 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ -5 \\ 0 \\ 7 \end{pmatrix}.$$

Solution.

We put the vectors as columns of a matrix A and find $|\det(A)|$. For this, we expand $\det(A)$ along the third row because it only has one nonzero entry.

$$\det(A) = 3(-1)^{3+1} \cdot \det\begin{pmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{pmatrix} = 3 \cdot 5(-1)^{1+3} \det\begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix} = 3(5)(1)(7-6) = 15.$$

(In the second step, we used the cofactor expansion along the first row since it had only one nonzero entry.)

The volume is $|\det(A)| = |15| = 15$.

4. Let $A = \begin{pmatrix} -1 & 1 \\ 1 & 7 \end{pmatrix}$, and define a transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = Ax$. Find the area of $T(S)$, if S is a triangle in $\mathbf{R}^2$ with area 2.

Solution.

$|\det(A)|\mathrm{Vol}(S) = |-7-1| \cdot 2 = 16.$

5. Let

$$A = \begin{pmatrix} 7 & 1 & 4 & 1 \\ -1 & 0 & 0 & 6 \\ 9 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 5 & 4 \\ 1 & -1 & -3 & 0 \\ -1 & 0 & 5 & 4 \\ 3 & -3 & -2 & 5 \end{pmatrix}$$

a) Compute $\det(A)$.

b) Compute $\det(B)$.

c) Compute $\det(AB)$.

d) Compute $\det(A^2B^{-1}AB^2)$.

Solution.

a) The second column has three zeros, so we expand by cofactors:

$$\det\begin{pmatrix} 7 & 1 & 4 & 1 \\ -1 & 0 & 0 & 6 \\ 9 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\det\begin{pmatrix} -1 & 0 & 6 \\ 9 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$$

Now we expand the second column by cofactors again:

$$\cdots = -2\det\begin{pmatrix} -1 & 6 \\ 0 & -1 \end{pmatrix} = (-2)(-1)(-1) = -2.$$

b) This is a complicated matrix without a lot of zeros, so we compute the determinant by row reduction. After one row swap and several row replacements, we reduce to the matrix

$$\begin{pmatrix} 1 & -1 & -3 & 0 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

The determinant of this matrix is -21, so the determinant of the original matrix is 21.

c) $\det(AB) = \det(A)\det(B) = (-2)(21) = -42.$

d) $\det(A^2B^{-1}AB^2) = \det(A)^2\det(B)^{-1}\det(A)\det(B)^2 = \det(A)^3\det(B) = (-2)^3(21) = -168.$

6. If A is a 3×3 matrix and $\det(A) = 1$, what is $\det(-2A)$?

Solution.

By determinant properties, scaling one row by c multiplies the determinant by c. When we take cA for an $n \times n$ matrix A, we are multiplying *each* row by c. This multiplies the determinant by c a total of n times.

Thus, if A is $n \times n$, then $\det(cA) = c^n \det(A)$. Here $n = 3$, so

$$\det(-2A) = (-2)^3 \det(A) = -8 \det(A) = -8.$$

7. **a)** Is there a real 2×2 matrix A that satisfies $A^4 = -I_2$? Either write such an A, or show that no such A exists.
(hint: think geometrically! The matrix $-I_2$ represents rotation by π radians).

b) Is there a real 3×3 matrix A that satisfies $A^4 = -I_3$? Either write such an A, or show that no such A exists.

Solution.

a) Yes. Just take A to be the matrix of counterclockwise rotation by $\frac{\pi}{4}$ radians:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then A^2 gives rotation c.c. by $\frac{\pi}{2}$ radians, A^3 gives rotation c.c. by $\frac{3\pi}{4}$ radians, and A^4 gives rotation c.c. by π radians, which has matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$.

b) No. If $A^4 = -I$ then

$$[\det(A)]^4 = \det(A^4) = \det(-I) = (-1)^3 = -1.$$

In other words, if $A^4 = -I$ then $[\det(A)]^4 = -1$, which is impossible since $\det(A)$ is a real number.

Similarly, $A^4 = -I$ is impossible if A is 5×5, 7×7, etc.