Supplemental problems: Chapter 4, Determinants

1. If A is an $n \times n$ matrix, is it necessarily true that $\det(-A) = -\det(A)$? Justify your answer.

2. Let A be an $n \times n$ matrix.

a) Using cofactor expansion, explain why $\det(A) = 0$ if A has a row or a column of zeros.

b) Using cofactor expansion, explain why $\det(A) = 0$ if A has adjacent identical columns.

3. Find the volume of the parallelepiped in $\mathbf{R}^4$ naturally determined by the vectors

$$\begin{pmatrix} 4 \\ 1 \\ 3 \\ 8 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 7 \\ 0 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ -5 \\ 0 \\ 7 \end{pmatrix}.$$

4. Let $A = \begin{pmatrix} -1 & 1 \\ 1 & 7 \end{pmatrix}$, and define a transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ by $T(x) = Ax$. Find the area of $T(S)$, if S is a triangle in $\mathbf{R}^2$ with area 2.

5. Let

$$A = \begin{pmatrix} 7 & 1 & 4 & 1 \\ -1 & 0 & 0 & 6 \\ 9 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 5 & 4 \\ 1 & -1 & -3 & 0 \\ -1 & 0 & 5 & 4 \\ 3 & -3 & -2 & 5 \end{pmatrix}$$

a) Compute $\det(A)$.

b) Compute $\det(B)$.

c) Compute $\det(AB)$.

d) Compute $\det(A^2B^{-1}AB^2)$.

6. If A is a 3×3 matrix and $\det(A) = 1$, what is $\det(-2A)$?

7. **a)** Is there a real 2×2 matrix A that satisfies $A^4 = -I_2$? Either write such an A, or show that no such A exists.
(hint: think geometrically! The matrix $-I_2$ represents rotation by π radians).

b) Is there a real 3×3 matrix A that satisfies $A^4 = -I_3$? Either write such an A, or show that no such A exists.