

Chapter 5

Eigenvalues and Eigenvectors

Section 5.1

Eigenvalues and Eigenvectors

A Biology Question

Motivation

In a population of rabbits:

1. half of the newborn rabbits survive their first year;
2. of those, half survive their second year;
3. their maximum life span is three years;
4. rabbits have 0, 6, 8 baby rabbits in their three years, respectively.

If you know the population one year, what is the population the next year?

f_n = first-year rabbits in year n

s_n = second-year rabbits in year n

t_n = third-year rabbits in year n

The rules say:

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ s_{n+1} \\ t_{n+1} \end{pmatrix}.$$

Let $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$ and $v_n = \begin{pmatrix} f_n \\ s_n \\ t_n \end{pmatrix}$. Then $A v_n = v_{n+1}$. ← difference equation

A Biology Question

Continued

If you know v_0 , what is v_{10} ?

$$v_{10} = Av_9 = AA v_8 = \dots = A^{10} v_0.$$

This makes it easy to compute examples by computer: [\[interactive\]](#)

v_0	v_{10}	v_{11}
$\begin{pmatrix} 3 \\ 7 \\ 9 \end{pmatrix}$	$\begin{pmatrix} 30189 \\ 7761 \\ 1844 \end{pmatrix}$	$\begin{pmatrix} 61316 \\ 15095 \\ 3881 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 9459 \\ 2434 \\ 577 \end{pmatrix}$	$\begin{pmatrix} 19222 \\ 4729 \\ 1217 \end{pmatrix}$
$\begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$	$\begin{pmatrix} 28856 \\ 7405 \\ 1765 \end{pmatrix}$	$\begin{pmatrix} 58550 \\ 14428 \\ 3703 \end{pmatrix}$

What do you notice about these numbers?

1. Eventually, each segment of the population doubles every year: $Av_n = v_{n+1} = 2v_n$.
2. The ratios get close to (16 : 4 : 1):

$$v_n = (\text{scalar}) \cdot \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}.$$

Translation: 2 is an eigenvalue, and $\begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$ is an eigenvector!

Definition

Let A be an $n \times n$ matrix.

Eigenvalues and eigenvectors are only for square matrices.

1. An **eigenvector** of A is a *nonzero* vector v in \mathbf{R}^n such that $Av = \lambda v$, for some λ in \mathbf{R} . In other words, Av is a multiple of v .
2. An **eigenvalue** of A is a number λ in \mathbf{R} such that the equation $Av = \lambda v$ has a *nontrivial* solution.

If $Av = \lambda v$ for $v \neq 0$, we say λ is the **eigenvalue for** v , and v is an **eigenvector for** λ .

Note: Eigenvectors are by definition nonzero. Eigenvalues may be equal to zero.

This is the most important definition in the course.

Verifying Eigenvectors

Example

$$A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \quad v = \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 16 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix} = 2v$$

Hence v is an eigenvector of A , with eigenvalue $\lambda = 2$.

Example

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Multiply:

$$Av = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4v$$

Hence v is an eigenvector of A , with eigenvalue $\lambda = 4$.

Poll

Which of the vectors

A. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ B. $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ C. $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ D. $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ E. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

are eigenvectors of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$?

What are the eigenvalues?

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

eigenvector with eigenvalue 2

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

eigenvector with eigenvalue 0

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

eigenvector with eigenvalue 0

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

not an eigenvector

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is never an eigenvector

Verifying Eigenvalues

Question: Is $\lambda = 3$ an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$?

In other words, does $Av = 3v$ have a nontrivial solution?

... does $Av - 3v = 0$ have a nontrivial solution?

... does $(A - 3I)v = 0$ have a nontrivial solution?

We know how to answer that! Row reduction!

$$A - 3I = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$$

Row reduce:

$$\begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

Parametric form: $x = -4y$; parametric vector form: $\begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -4 \\ 1 \end{pmatrix}$.

Does there exist an eigenvector with eigenvalue $\lambda = 3$? Yes! Any nonzero multiple of $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$. Check:

$$\begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -12 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} -4 \\ 1 \end{pmatrix}. \quad \checkmark$$

Definition

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A . The λ -**eigenspace** of A is the set of all eigenvectors of A with eigenvalue λ , plus the zero vector:

$$\begin{aligned}\lambda\text{-eigenspace} &= \{v \text{ in } \mathbf{R}^n \mid Av = \lambda v\} \\ &= \{v \text{ in } \mathbf{R}^n \mid (A - \lambda I)v = 0\} \\ &= \text{Nul}(A - \lambda I).\end{aligned}$$

Since the λ -eigenspace is a null space, it is a *subspace* of \mathbf{R}^n .

How do you find a basis for the λ -eigenspace? Parametric vector form!

Eigenspaces

Example

Find a basis for the 3-eigenspace of

$$A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}.$$

We have to solve the matrix equation $A - 3I_2 = 0$.

$$A - 3I_2 = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -4 \\ -1 & -4 \end{pmatrix}$$

$$\begin{array}{l} \text{RREF} \\ \rightsquigarrow \end{array} \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

$$\begin{array}{l} \text{parametric form} \\ \rightsquigarrow \end{array} x = -4y$$

$$\begin{array}{l} \text{parametric vector form} \\ \rightsquigarrow \end{array} \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

$$\begin{array}{l} \text{basis} \\ \rightsquigarrow \end{array} \left\{ \begin{pmatrix} -4 \\ 1 \end{pmatrix} \right\}.$$

Eigenspaces

Example

Find a basis for the 2-eigenspace of

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

$$A - 2I = \begin{pmatrix} 3/2 & 0 & 3 \\ -3/2 & 0 & -3 \\ -3/2 & 0 & -3 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{parametric form}} x = -2z$$

$$\xrightarrow{\text{parametric vector form}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\xrightarrow{\text{basis}} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Eigenspaces

Example

Find a basis for the $\frac{1}{2}$ -eigenspace of

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

$$A - \frac{1}{2}I = \begin{pmatrix} 3 & 0 & 3 \\ -3/2 & 3/2 & -3 \\ -3/2 & 0 & -3/2 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{parametric form}} \begin{cases} x = -z \\ y = z \end{cases}$$

$$\xrightarrow{\text{parametric vector form}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\xrightarrow{\text{basis}} \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Eigenspaces

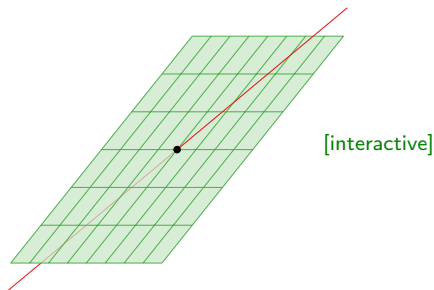
Example: picture

$$A = \begin{pmatrix} 7/2 & 0 & 3 \\ -3/2 & 2 & -3 \\ -3/2 & 0 & -1 \end{pmatrix}.$$

We computed bases for the 2-eigenspace and the $1/2$ -eigenspace:

$$\text{2-eigenspace: } \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \frac{1}{2}\text{-eigenspace: } \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Hence the 2-eigenspace is a plane and the $1/2$ -eigenspace is a line.



Let A be an $n \times n$ matrix and let λ be a number.

1. λ is an eigenvalue of A if and only if $(A - \lambda I)x = 0$ has a nontrivial solution, if and only if $\text{Nul}(A - \lambda I) \neq \{0\}$.
2. In this case, finding a basis for the λ -eigenspace of A means finding a basis for $\text{Nul}(A - \lambda I)$ as usual, i.e. by finding the parametric vector form for the general solution to $(A - \lambda I)x = 0$.
3. The eigenvectors with eigenvalue λ are the nonzero elements of $\text{Nul}(A - \lambda I)$, i.e. the nontrivial solutions to $(A - \lambda I)x = 0$.

The Eigenvalues of a Triangular Matrix are the Diagonal Entries

We've seen that finding eigenvectors for a given eigenvalue is a row reduction problem.

Finding all of the eigenvalues of a matrix *is not a row reduction problem!* We'll see how to do it in general next time. For now:

Fact: The eigenvalues of a triangular matrix are the diagonal entries.

Why? $\text{Nul}(A - \lambda I) \neq \{0\}$ if and only if $A - \lambda I$ is not invertible, if and only if $\det(A - \lambda I) = 0$.

$$\begin{pmatrix} 3 & 4 & 1 & 2 \\ 0 & -1 & -2 & 7 \\ 0 & 0 & 8 & 12 \\ 0 & 0 & 0 & -3 \end{pmatrix} - \lambda I_4 = \begin{pmatrix} 3 - \lambda & 4 & 1 & 2 \\ 0 & -1 - \lambda & -2 & 7 \\ 0 & 0 & 8 - \lambda & 12 \\ 0 & 0 & 0 & -3 - \lambda \end{pmatrix}.$$

The determinant is $(3 - \lambda)(-1 - \lambda)(8 - \lambda)(-3 - \lambda)$, which is zero exactly when $\lambda = 3, -1, 8,$ or -3 .

A Matrix is Invertible if and only if Zero is not an Eigenvalue

Fact: A is invertible if and only if 0 is not an eigenvalue of A .

Why?

0 is an eigenvalue of $A \iff Ax = 0x$ has a nontrivial solution

$\iff Ax = 0$ has a nontrivial solution

$\iff A$ is not invertible.

invertible matrix theorem



Eigenvectors with Distinct Eigenvalues are Linearly Independent

Fact: If v_1, v_2, \dots, v_k are eigenvectors of A with *distinct* eigenvalues $\lambda_1, \dots, \lambda_k$, then $\{v_1, v_2, \dots, v_k\}$ is linearly independent.

Why? If $k = 2$, this says v_2 can't lie on the line through v_1 .

But the line through v_1 is contained in the λ_1 -eigenspace, and v_2 does not have eigenvalue λ_1 .

In general: see §5.1 (or work it out for yourself; it's not too hard).

Consequence: An $n \times n$ matrix has at most n distinct eigenvalues.

The Invertible Matrix Theorem

Addenda

We have a couple of new ways of saying “ A is invertible” now:

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix, and let $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ be the linear transformation $T(x) = Ax$. The following statements are equivalent.

1. A is invertible.
2. T is invertible.
3. The reduced row echelon form of A is I_n .
4. A has n pivots.
5. $Ax = 0$ has no solutions other than the trivial one.
6. $\text{Nul}(A) = \{0\}$.
7. $\text{nullity}(A) = 0$.
8. The columns of A are linearly independent.
9. The columns of A form a basis for \mathbf{R}^n .
10. T is one-to-one.
11. $Ax = b$ is consistent for all b in \mathbf{R}^n .
12. $Ax = b$ has a unique solution for each b in \mathbf{R}^n .
13. The columns of A span \mathbf{R}^n .
14. $\text{Col } A = \mathbf{R}^m$.
15. $\dim \text{Col } A = m$.
16. $\text{rank } A = m$.
17. T is onto.
18. There exists a matrix B such that $AB = I_n$.
19. There exists a matrix B such that $BA = I_n$.
20. The determinant of A is *not* equal to zero.
21. The number 0 is *not* an eigenvalue of A .

Summary

- ▶ **Eigenvectors** and **eigenvalues** are the most important concepts in this course.
- ▶ Eigenvectors are by definition nonzero; eigenvalues may be zero.
- ▶ The eigenvalues of a triangular matrix are the diagonal entries.
- ▶ A matrix is invertible if and only if zero is not an eigenvalue.
- ▶ Eigenvectors with distinct eigenvalues are linearly independent.
- ▶ The λ -eigenspace is the set of all λ -eigenvectors, plus the zero vector.
- ▶ You can compute a basis for the λ -eigenspace by finding the parametric vector form of the solutions of $(A - \lambda I_n)x = 0$.