Math 1553 Worksheet §6.1, §6.2

Solutions

1. True/False

(1) If u is in subspace W, and u is also in $W^{\perp}$, then $u = 0$.
(2) If y is in subspace W, the orthogonal projection of y onto W is y.
(3) If x is orthogonal to v and w, then x is also orthogonal to $v - w$.

Solution.

(1) TRUE: $u \cdot u = ||u||^{1/2} = 0$ only when $u = 0$.
(2) TRUE: y is decomposed into the elements that form a basis for W, so they could be used to give a unique representation for y.
(3) TRUE: $v - w$ is on the same plane spanned by v and w.

2. Give examples

(1) two linearly independent vectors that are orthogonal to $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.
(2) a subspace of $\mathbf{R}^3$, S, such that $\dim(S^{\perp}) = 2$.

Solution.

(1) Two linearly independent vectors orthogonal to the first vector can be found, for example, by setting

$$u = \begin{pmatrix} 1 \\ 0 \\ a \end{pmatrix}, v = \begin{pmatrix} 0 \\ 1 \\ b \end{pmatrix}$$

Taking dot products and solving for a and b gives the vectors we need. Students will likely get the answer by inspection in a way that takes advantage of the second element being zero, but you can ask how we might approach a more general problem.

(2) A few examples:

- The column space of $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
- The span of any vector in $\mathbb{R}^3$.
- The null space of a 3×3 matrix that has two pivots.

3. **a)** Compute dot product of every pair of two vectors from $u = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$ and $w = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ -1 \end{pmatrix}$.

b) What are the eigenvalues and eigenvectors of the 3×3 matrix $A = vv^T$?

c) What is the column space and null space of the matrix $A = vv^T$?

Solution.

a) If we compute the dot products, we find $u \cdot v = u \cdot w = v \cdot w = 0$. They are mutually orthogonal to each other. In fact, these three vectors form an orthogonal basis for $\mathbf{R}^3$.

b) Notice $Av = vv^T v = v(v^T v) = v$ gives us an eigenvalue $\lambda_1 = 1$, eigenvector v. Similarly $Au = vv^T u = v(v^T u) = 0$ and $Aw = vv^T w = v(v^T w) = 0$ gives us eigenvalue $\lambda_2 = \lambda_3 = 0$, eigenvectors u, w. Since A is 3×3, there is no other eigenvalues.

c) One way to do this is to compute explicitly

$$A = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then one can compute $\text{Col}A = \text{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right\}$, and $\text{Nul}A = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$.

Another way to do this without computing A, is by looking at the geometry through eigenvectors. This matrix A preserves any vector in the space $V = \text{Span}\{v\}$. Moreover A projects vecors in $V^\perp = \text{Span}\{u, w\}$ to the 0 vector. So the geometry suggests, A is a projection matrix which has $\text{Nul}A = V^\perp = \text{Span}\{u, w\}$, and $\text{Col}A = V = \text{Span}\{v\}$.