Math 1553 Worksheet §6.1 - §6.5

Solutions

1. True/False

(1) If u is in subspace W, and u is also in $W^\perp$, then $u = 0$.
(2) If y is in a subspace W, the orthogonal projection of y onto $W^\perp$ is 0.
(3) If x is orthogonal to v and w, then x is also orthogonal to $v - w$.

Solution.

(1) TRUE: Such a vector u would be orthogonal to itself, so $u \cdot u = ||u||^2 = 0$. Therefore, u has length 0, so $u = 0$.
(2) TRUE: y is in W, so $y \perp W^\perp$. Its orthogonal projection onto W is y and orthogonal projection onto $W^\perp$ is 0. In fact y has orthogonal decomposition $y = y + 0$, where y is in W and 0 is in $W^\perp$.
(3) TRUE: By properties of the dot product, if x is orthogonal to v and w then x is orthogonal to everything in Span$\{v, w\}$ (which includes $v - w$).

2. **a)** Find the standard matrix B for proj_L, where $L = \text{Span}\left\{\begin{pmatrix}1\\1\\-1\end{pmatrix}\right\}$.

b) What are the eigenvalues of B? Is B is diagonalizable? If so, find an invertible C and diagonal D so that $B = CDC^{-1}$?

c) Describe the column space and null space of the matrix B in terms of L.

Solution.

a) We use the formula $B = \dfrac{1}{u \cdot u} uu^T$ where $u = \begin{pmatrix}1\\1\\-1\end{pmatrix}$ (this is the formula $B = A(A^TA)^{-1}A^T$ when "A" is just the single vector u).

$$B = \frac{1}{1(1)+1(1)+(-1)(-1)}\begin{pmatrix}1\\1\\-1\end{pmatrix}\begin{pmatrix}1 & 1 & -1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1 & 1 & -1\\1 & 1 & -1\\-1 & -1 & 1\end{pmatrix}$$

$$\implies B = \frac{1}{3}\begin{pmatrix}1 & 1 & -1\\1 & 1 & -1\\-1 & -1 & 1\end{pmatrix}.$$

b) $Bx = x$ for every x in L, and $Bx = 0$ for every x in $L^\perp$, so B has two eigenvalues: $\lambda_1 = 1$ with algebraic (and geometric) multiplicity 1, $\lambda_2 = 0$ with algebraic (and geometric) multiplicity 2. Here $v_1 = \begin{pmatrix}1\\1\\-1\end{pmatrix}$ is an eigenvector for $\lambda_1 = 1$, whereas $v_2 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$ and $v_3 = \begin{pmatrix}1\\0\\1\end{pmatrix}$ are eigenvectors for $\lambda_2 = 0$.

Therefore

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^{-1}$$

c) $\mathrm{Col}(B) = L$ and $\mathrm{Nul}(B) = L^{\perp}$.

3. $y = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}, \quad u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

(1) Determine whether u_1 and u_2
   (a) are linearly independent
   (b) are orthogonal
   (c) span $\mathbf{R}^3$
(2) Is y in $W = \mathrm{Span}\{u_1, u_2\}$?
(3) Compute the vector w that most closely approximates y within W.
(4) Construct a vector, z, that is in $W^{\perp}$.
(5) Make a rough sketch of W, y, w, and z.

Solution.

(1) A quick check shows that the vectors u_1 and u_2 are orthogonal and linearly independent, so $\mathrm{Span}\{u_1, u_2\}$ is a plane in $\mathbf{R}^3$, but is not all of $\mathbf{R}^3$.
(2) By inspection, y is not in the span because it has a non-zero x_3 component.
(3) The vector w is $\mathrm{proj}_W y$. The orthogonal projection of y onto W is calculated in the usual way.

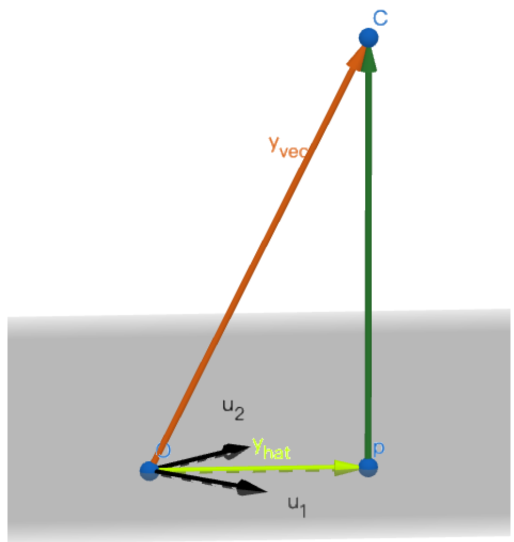
$$A^T A v = A^T b$$

$$A^T A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad A^T b = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \qquad \text{so} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} v = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \qquad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$w = Av = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

Another quick way to do this problem is note that W is the xy-plane of $\mathbf{R}^3$, so the projection of $\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$ onto W is just $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

(4) One vector in $W^{\perp}$ is $z = y - \mathrm{proj}_W y = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$.
(5) Here `https://www.geogebra.org/calculator/` is a picture you can play with. The vector w is labeled "$\mathrm{y}_{\mathrm{hat}}$" in the drawing.



4. **a)** Find the least squares solution $\widehat{x}$ to $Ax = e_1$, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$.

b) Find the best fit line $y = Ax + B$ through the points $(0,0)$, $(1,8)$, $(3,8)$, and $(4,20)$.

c) Set up an equation to find the best fit parabola $y = Ax^2 + Bx + C$ through the points $(0,0)$, $(1,8)$, $(3,8)$, and $(4,20)$.

Solution.

a) We need to solve the equation $A^T A\widehat{x} = A^T e_1$. We compute:

$$A^T A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$A^T e_1 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Now we form the augmented matrix:

$$\left(\begin{array}{cc|c} 2 & 0 & 1 \\ 0 & 3 & 1 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 1/3 \end{array}\right) \implies \widehat{x} = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix}.$$

b) We want to find a least squares solution to the system of linear equations

$$\begin{aligned} 0 &= A(0) + B \\ 8 &= A(1) + B \\ 8 &= A(3) + B \\ 20 &= A(4) + B \end{aligned} \qquad \iff \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}.$$

We compute

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 26 & 8 \\ 8 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 112 \\ 36 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 26 & 8 & 112 \\ 8 & 4 & 36 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right).$$

Hence the least squares solution is $A = 4$ and $B = 1$, so the best fit line is $y = 4x + 1$.

c) We want to find a least squares solution to the system of linear equations

$$\begin{aligned} 0 &= A(0^2) + B(0) + C \\ 8 &= A(1^2) + B(1) + C \\ 8 &= A(3^2) + B(3) + C \\ 20 &= A(4^2) + B(4) + C \end{aligned} \qquad \iff \qquad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}.$$

We compute

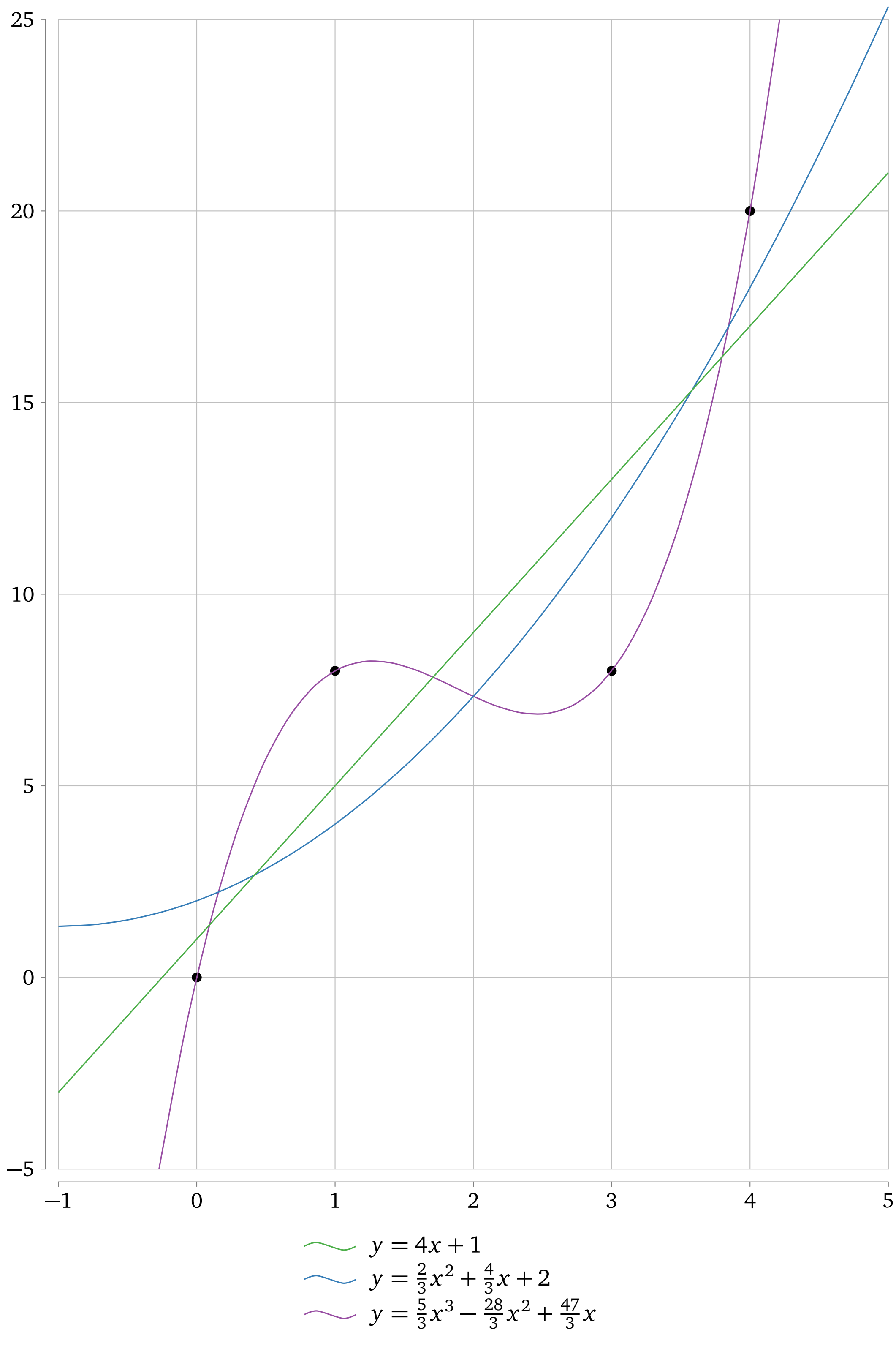
$$\begin{pmatrix} 0 & 1 & 9 & 16 \\ 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 338 & 92 & 26 \\ 92 & 26 & 8 \\ 26 & 8 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 9 & 16 \\ 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 400 \\ 112 \\ 36 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 338 & 92 & 26 & 400 \\ 92 & 26 & 8 & 112 \\ 26 & 8 & 4 & 36 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 4/3 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

Hence the least squares solution is $A = 2/3$, $B = 4/3$, and $C = 2$, so the best fit quadratic is $y = \frac{2}{3}x^2 + \frac{4}{3}x + 2$.

There is a picture on the next page. The "best fit cubic" would be the cubic $y = \frac{5}{3}x^3 - \frac{28}{3}x^2 + \frac{47}{3}x$, which actually passes through all four points. One can fit the points with even higher order polynomials.


25
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2
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$y = 4x + 1$
$y = \frac{2}{3}x^2 + \frac{4}{3}x + 2$
$y = \frac{5}{3}x^3 - \frac{28}{3}x^2 + \frac{47}{3}x$