Math 1553 Worksheet §5.1 - §5.4

Solutions

1. True or false. If the statement is always true, answer true and justify why it is true. Otherwise, answer false and give an example that shows it is false. In every case, assume that A is an $n \times n$ matrix.

a) The entries on the main diagonal of A are the eigenvalues of A.

b) The number λ is an eigenvalue of A if and only if there is a nonzero solution to the equation $(A - \lambda I)x = 0$.

c) To find the eigenvectors of A, we reduce the matrix A to row echelon form.

d) If A is invertible and 2 is an eigenvalue of A, then $\frac{1}{2}$ is an eigenvalue of A^{-1}.

e) If Nul(A) has dimension at least 1, then Nul(A) is the eigenspace of A corresponding to the eigenvalue 0.

Solution.

a) False. This is true if A is triangular, but not in general.

For example, if $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ then the diagonal entries are 2 and 0 but the only eigenvalue is $\lambda = 1$, since solving the characteristic equation gives us

$$(2-\lambda)(-\lambda) - (1)(-1) = 0 \qquad \lambda^2 - 2\lambda + 1 = 0 \qquad (\lambda - 1)^2 = 0 \qquad \lambda = 1.$$

b) True.

$$(A - \lambda I)x = 0 \iff Ax - \lambda x = 0 \iff Ax = \lambda x.$$

Therefore, $(A - \lambda I)x = 0$ has a nonzero solution if and only if $Ax = \lambda x$ has a nonzero solution, which is to say that λ is an eigenvalue of A.

c) False. The RREF of A will only compute the eigenvectors with eigenvalue zero, or will tell us that zero is not an eigenvalue. To get the eigenvectors corresponding to an eigenvalue λ, we put $A - \lambda I$ into RREF and write the solutions of $\left(A - \lambda I \mid 0\right)$ in parametric vector form.

d) True. Let v be an eigenvector corresponding to the eigenvalue 2.

$$Av = 2v \implies A^{-1}Av = A^{-1}(2v) \implies v = 2A^{-1}v \implies \frac{1}{2}v = A^{-1}v.$$

Therefore, v is an eigenvector of A^{-1} corresponding to the eigenvalue $\frac{1}{2}$.

e) True. For every v in Nul A, we have $Av = 0v$. If $v \neq 0$, this is exactly the definition of v being an eigenvector corresponding to the eigenvalue 0. If Nul$A$ has dimension at least 1, then infinitely many nonzero vectors satisfy $Av = 0$, so 0 is an eigenvalue of A (and every nonzero vector v satisfying $Av = 0$ is an eigenvector of A) and Nul A is the 0-eigenspace of A.

2. Suppose A is an $n \times n$ matrix satisfying $A^2 = 0$. Find all eigenvalues of A. Justify your answer.

Solution.

If λ is an eigenvalue of A and $v \neq 0$ is a corresponding eigenvector, then

$$Av = \lambda v \implies A(Av) = A\lambda v \implies A^2 v = \lambda(Av) \implies 0 = \lambda(\lambda v) \implies 0 = \lambda^2 v.$$

Since $v \neq 0$ this means $\lambda^2 = 0$, so $\lambda = 0$. This shows that 0 is the only possible eigenvalue of A.

On the other hand, $\det(A) = 0$ since $(\det(A))^2 = \det(A^2) = \det(0) = 0$, so 0 must be an eigenvalue of A. Therefore, the only eigenvalue of A is 0.

3. Answer yes, no, or maybe. Justify your answers. In each case, A is a matrix whose entries are real numbers.

a) Suppose $A = \begin{pmatrix} 3 & 0 & 0 \\ 5 & 1 & 0 \\ -10 & 4 & 7 \end{pmatrix}$. Then the characteristic polynomial of A is

$$\det(A - \lambda I) = (3 - \lambda)(1 - \lambda)(7 - \lambda).$$

b) If A is a 3×3 matrix with characteristic polynomial $-\lambda(\lambda - 5)^2$, then the 5-eigenspace is 2-dimensional.

c) If A is an invertible 2×2 matrix, then A is diagonalizable.

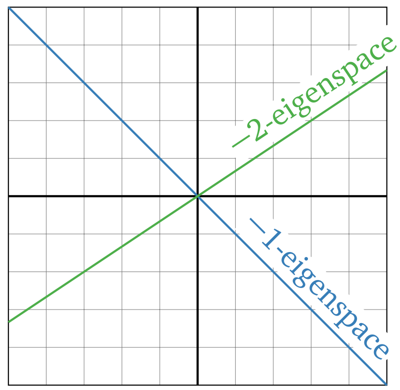
Solution.

a) Yes. Since $A - \lambda I$ is triangular, its determinant is the product of its diagonal entries.

b) Maybe. The geometric multiplicity of $\lambda = 5$ can be 1 or 2. For example, the matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 2-dimensional, whereas the matrix $\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 1-dimensional. Both matrices have characteristic polynomial $-\lambda(5 - \lambda)^2$.

c) Maybe. The identity matrix is invertible and diagonalizable, but the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible but not diagonalizable.

4. The eigenspaces of some 2×2 matrix A are drawn below. Write an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$. Can you find another pair of C and D that does the same?



Solution.

We choose D to be a diagonal matrix whose entries are the eigenvalues of A, and C a matrix whose columns are corresponding eigenvectors (written in the same order).

The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -2$.
The (-1)-eigenspace is spanned by $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
The (-2)-eigenspace is spanned by $v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.
Therefore, we can choose $C = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$.

There are many possibilities for C and D.
For example, since $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} = \mathrm{Span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$, we could have chosen $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ instead, so that

$$C = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}.$$

Alternatively, we could have rearranged the order of the diagonal entries of D and taken care to use the corresponding order in the columns of C:

$$C = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Regardless, if you write any correct answers for C and D and go the extra step of carrying out the computation, you will obtain

$$A = CDC^{-1} = -\frac{1}{5}\begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}.$$