Section 3.4

Matrix Multiplication

Motivation

Recall: we can turn any system of linear equations into a matrix equation

$$Ax = b.$$

This notation is suggestive. Can we solve the equation by "dividing by A"?

$$x \stackrel{??}{=} \frac{b}{A}$$

Answer: Sometimes, but you have to know what you're doing.

Today we'll study *matrix algebra*: adding and multiplying matrices.

These are not so hard to do. The important thing to understand today is the relationship between *matrix multiplication* and *composition of transformations*.

More Notation for Matrices

Let A be an $m \times n$ matrix.

We write a_{ij} for the entry in the ith row and the jth column. It is called the **ijth entry** of the matrix.

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

ith row

jth column

The entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **diagonal entries**; they form the **main diagonal** of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

A **diagonal matrix** is a *square* matrix whose only nonzero entries are on the main diagonal.

$$\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

The $n \times n$ **identity matrix** I_n is the diagonal matrix with all diagonal entries equal to 1. It is special because $I_n v = v$ for *all* v in $\mathbf{R}^n$.

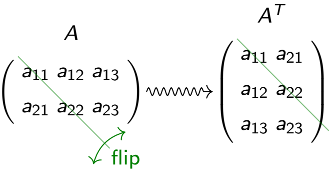
$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

More Notation for Matrices
Continued

The **zero matrix** (of size $m \times n$) is the $m \times n$ matrix 0 with all zero entries.

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose rows are the columns of A. In other words, the ij entry of A^T is a_{ji}.



Matrix Multiplication

Beware: matrix multiplication is more subtle than addition and scalar multiplication.

must be equal

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix with columns $v_1, v_2, \ldots, v_p$:

$$B = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{pmatrix}.$$

The **product** AB is the $m \times p$ matrix with columns $Av_1, Av_2, \ldots, Av_p$:

The equality is a definition

$$AB \overset{\text{def}}{=} \begin{pmatrix} | & | & & | \\ Av_1 & Av_2 & \cdots & Av_p \\ | & | & & | \end{pmatrix}.$$

In order for $Av_1, Av_2, \ldots, Av_p$ to make sense, the number of columns of A has to be the same as the number of rows of B. Note the sizes of the product!

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 14 & -10 \\ 32 & -28 \end{pmatrix}$$

The Row-Column Rule for Matrix Multiplication

Recall: A row vector of length n times a column vector of length n is a scalar:

$$\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n.$$

Another way of multiplying a matrix by a vector is:

$$Ax = \begin{pmatrix} \text{---}\ r_1\ \text{---} \\ \vdots \\ \text{---}\ r_m\ \text{---} \end{pmatrix} x = \begin{pmatrix} r_1 x \\ \vdots \\ r_m x \end{pmatrix}.$$

On the other hand, you multiply two matrices by

$$AB = A \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Ac_1 & \cdots & Ac_p \\ | & & | \end{pmatrix}.$$

It follows that

$$AB = \begin{pmatrix} \text{---}\ r_1\ \text{---} \\ \vdots \\ \text{---}\ r_m\ \text{---} \end{pmatrix} \begin{pmatrix} | & & | \\ c_1 & \cdots & c_p \\ | & & | \end{pmatrix} = \begin{pmatrix} r_1 c_1 & r_1 c_2 & \cdots & r_1 c_p \\ r_2 c_1 & r_2 c_2 & \cdots & r_2 c_p \\ \vdots & \vdots & & \vdots \\ r_m c_1 & r_m c_2 & \cdots & r_m c_p \end{pmatrix}$$

The Row-Column Rule for Matrix Multiplication

The ij entry of $C = AB$ is the ith row of A times the jth column of B:

$$c_{ij} = (AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

This is how everybody on the planet actually computes AB. Diagram ($AB = C$):

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ik} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mk} & \cdots & a_{mn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kp} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mp} \end{pmatrix}$$

ith row | jth column | ij entry

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 2 + 3\cdot 3 & \square \\ \square & \square \end{pmatrix} = \begin{pmatrix} 14 & \square \\ \square & \square \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} \square & \square \\ 4\cdot 1 + 5\cdot 2 + 6\cdot 3 & \square \end{pmatrix} = \begin{pmatrix} \square & \square \\ 32 & \square \end{pmatrix}$$

Composition of Transformations

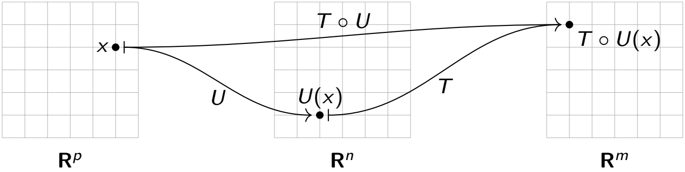
Why is this the correct definition of matrix multiplication?

Definition

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ and $U\colon \mathbf{R}^p \to \mathbf{R}^n$ be transformations. The **composition** is the transformation

$$T \circ U\colon \mathbf{R}^p \to \mathbf{R}^m \qquad \text{defined by} \qquad T \circ U(x) = T(U(x)).$$

This makes sense because $U(x)$ (the output of U) is in $\mathbf{R}^n$, which is the domain of T (the inputs of T). [interactive]



Fact: If T and U are linear then so is $T \circ U$.

Guess: If A is the matrix for T, and B is the matrix for U, what is the matrix for $T \circ U$?

Composition of Linear Transformations

Let $T\colon \mathbf{R}^n \to \mathbf{R}^m$ and $U\colon \mathbf{R}^p \to \mathbf{R}^n$ be *linear* transformations. Let A and B be their matrices:

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix} \quad B = \begin{pmatrix} | & | & & | \\ U(e_1) & U(e_2) & \cdots & U(e_p) \\ | & | & & | \end{pmatrix}$$

Question

What is the matrix for $T \circ U$?

We find the matrix for $T \circ U$ by plugging in the unit coordinate vectors:

$U(e_1) = Be_1$ is the first column of B

the first column of AB is $A(Be_1)$

$$T \circ U(e_1) = T(U(e_1)) = T(Be_1) = A(Be_1) = (AB)e_1.$$

For any other i, the same works:

$$T \circ U(e_i) = T(U(e_i)) = T(Be_i) = A(Be_i) = (AB)e_i.$$

This says that the ith column of the matrix for $T \circ U$ is the ith column of AB.

The matrix of the composition is the product of the matrices!

We can also add and scalar multiply linear transformations:

$$T, U\colon \mathbf{R}^n \to \mathbf{R}^m \quad \rightsquigarrow \quad T + U\colon \mathbf{R}^n \to \mathbf{R}^m \qquad (T + U)(x) = T(x) + U(x).$$

In other words, add transformations "pointwise".

$$T\colon \mathbf{R}^n \to \mathbf{R}^m \qquad c \text{ in } \mathbf{R} \quad \rightsquigarrow \quad cT\colon \mathbf{R}^n \to \mathbf{R}^m \qquad (cT)(x) = c \cdot T(x).$$

In other words, scalar-multiply a transformation "pointwise".

The next slide describes these operations in terms of matrix algebra.

Addition and Scalar Multiplication for Matrices

You add two matrices component by component, like with vectors.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{pmatrix}$$

Note you can only add two matrices *of the same size*.

You multiply a matrix by a scalar by multiplying each component, like with vectors.

$$c \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \end{pmatrix}.$$

These satisfy the expected rules, like with vectors:

$$\begin{aligned} A+B &= B+A & (A+B)+C &= A+(B+C) \\ c(A+B) &= cA+cB & (c+d)A &= cA+dA \\ (cd)A &= c(dA) & A+0 &= A \end{aligned}$$

If linear transformations T and U have matrices A and B, respectively:

- $T+U$ has matrix $A+B$.
- cT has matrix cA.

Let $T\colon \mathbf{R}^3 \to \mathbf{R}^2$ and $U\colon \mathbf{R}^2 \to \mathbf{R}^3$ be the matrix transformations

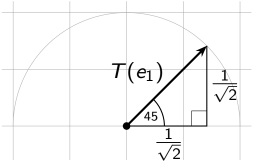
$$T(x) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} x \qquad U(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} x.$$

Then the matrix for $T \circ U$ is

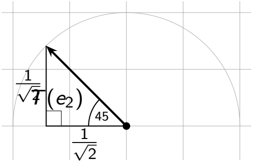
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

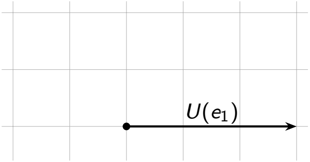
[interactive]

Let $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ be rotation by 45°, and let $U\colon \mathbf{R}^2 \to \mathbf{R}^2$ scale the x-coordinate by 1.5. Let's compute their standard matrices A and B:

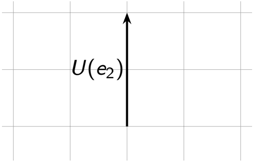


$$T(e_1) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad T(e_2) = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$$





$$U(e_1) = \begin{pmatrix}1.5\\0\end{pmatrix} \qquad U(e_2) = \begin{pmatrix}0\\1\end{pmatrix}$$



$$\implies \quad A = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix} \qquad B = \begin{pmatrix}1.5 & 0\\0 & 1\end{pmatrix}$$

Composition of Linear Transformations

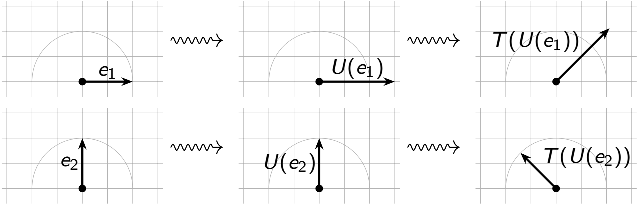
Another example, continued

So the matrix C for $T \circ U$ is

$$C = AB = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1.5 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \left(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1.5 \\ 0 \end{pmatrix} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1.5 & -1 \\ 1.5 & 1 \end{pmatrix}.$$

Check: [interactive: e_1] [interactive: e_2]

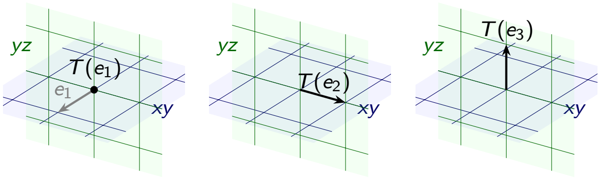


$$T \circ U(e_1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix}$$

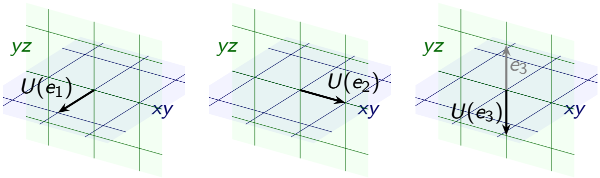
$$T \circ U(e_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\implies C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1.5 & -1 \\ 1.5 & 1 \end{pmatrix} \quad ✔$$

Let $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ be projection onto the yz-plane, and let $U\colon \mathbf{R}^3 \to \mathbf{R}^3$ be reflection over the xy-plane. Let's compute their standard matrices A and B:



$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Composition of Linear Transformations

Another example, continued

So the matrix C for $T \circ U$ is

$$C = AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Check: we did this last time ✔

[interactive: e_1] [interactive: e_2] [interactive: e_3]

Poll

Poll

Do there exist *nonzero* matrices A and B with $AB = 0$?

Yes! Here's an example:

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Properties of Matrix Multiplication

Mostly matrix multiplication works like you'd expect. Suppose A has size $m \times n$, and that the other matrices below have the right size to make multiplication work.

$$\begin{aligned} A(BC) &= (AB)C \\ (B+C)A &= BA + CA \\ c(AB) &= A(cB) \\ AI_n &= A \end{aligned} \qquad \begin{aligned} A(B+C) &= (AB + AC) \\ c(AB) &= (cA)B \\ I_m A &= A \end{aligned}$$

Most of these are easy to verify.

Associativity is $A(BC) = (AB)C$. It is a pain to verify using the row-column rule! Much easier: use associativity of linear transformations:

$$S \circ (T \circ U) = (S \circ T) \circ U.$$

This is a good example of an instance where having a conceptual viewpoint saves you a lot of work.

Recommended: Try to verify all of them on your own.

Warnings!

- AB is usually not equal to BA.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$$

In fact, AB may be defined when BA is not.

- $AB = AC$ does not imply $B = C$, even if $A \neq 0$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 5 & 6 \end{pmatrix}$$

- $AB = 0$ does not imply $A = 0$ or $B = 0$.

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Powers of a Matrix

Suppose A is a *square* matrix.

Then $A \cdot A$ makes sense, and has the same size.

Then $A \cdot (A \cdot A)$ also makes sense and has the same size.

Definition

Let n be a positive whole number and let A be a square matrix. The **nth power** of A is the product

$$A^n = \underbrace{A \cdot A \cdot \cdots \cdot A}_{n \text{ times}}$$

Example

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$\cdots \quad A^n = \begin{pmatrix} 1 & n-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

Summary

- The product of an $m \times n$ matrix and an $n \times p$ matrix is an $m \times p$ matrix. I showed you two ways of computing the product.
- Composition of linear transformations corresponds to multiplication of matrices.
- You have to be careful when multiplying matrices together, because things like commutativity and cancellation fail.
- You can take powers of square matrices.