Math 1553 Worksheet §5.4, 5.5, 5.6

1. True or false. Justify your answer.

a) A 3×3 matrix A can have a non-real complex eigenvalue with multiplicity 2.

b) It is possible for a 2×2 stochastic matrix to have $-i/2$ as an eigenvalue.

Solution.

a) No. If c is a (non-real) complex eigenvalue with multiplicity 2, then its conjugate $\bar{c}$ is an eigenvalue with multiplicity 2 since complex eigenvalues always occur in conjugate pairs. This would mean A has a characteristic polynomial of degree 4 or more, which is impossible since A is 3×3.

b) No. The matrix must have $\lambda = 1$ as an eigenvalue since it is stochastic, but if $\lambda = -i/2$ is an eigenvalue then so is $\lambda = i/2$, which is impossible since a 2×2 matrix cannot have more than two eigenvalues.

2. Let $A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}^{-1}$, and let $x = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. What happens to $A^n x$ as n gets very large?

Solution.

We are given diagonalization of A, which gives us the eigenvalues and eigenvectors.

$$\begin{aligned} A^n x &= A^n\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right) = A^n \begin{pmatrix} 2 \\ -1 \end{pmatrix} + A^n \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= 1^n \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \left(\frac{1}{2}\right)^n \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} \frac{3}{2^n} \\ \frac{1}{2^n} \end{pmatrix}. \end{aligned}$$

As n gets very large, the entries in the second vector above approach zero, so $A^n x$ approaches $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. For example, for $n = 15$,

$$A^{15} x \approx \begin{pmatrix} 2.00009 \\ -0.999969 \end{pmatrix}.$$

3. Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Find all eigenvalues of A. For each eigenvalue, find an associated eigenvector.

Solution.

The characteristic polynomial is

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 5$$

$$\lambda^2 - 2\lambda + 5 = 0 \iff \lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

For the eigenvalue $\lambda = 1-2i$, we use the shortcut trick you may have seen in class: the first row $\begin{pmatrix} a & b \end{pmatrix}$ of $A - \lambda I$ will lead to an eigenvector $\begin{pmatrix} -b \\ a \end{pmatrix}$ (or equivalently, $\begin{pmatrix} b \\ -a \end{pmatrix}$ if you prefer).

$$\left(A-(1-2i)I \mid 0\right) = \left(\begin{array}{cc|c} 2i & 2 & 0 \\ (*) & (*) & 0 \end{array}\right) \implies v = \begin{pmatrix} -2 \\ 2i \end{pmatrix}.$$

From the correspondence between conjugate eigenvalues and their eigenvectors, we know (without doing any additional work!) that for the eigenvalue $\lambda = 1+2i$, a corresponding eigenvector is $w = \overline{v} = \begin{pmatrix} -2 \\ -2i \end{pmatrix}$.

If you used row-reduction for finding eigenvectors, you would find $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$ as an eigenvector for eigenvalue $1-2i$, and $w = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ as an eigenvector for eigenvalue $1+2i$.

4. A video game offers participants the chance to play as one of three characters: Archer, Barbarian, or Cleric. The game has 72 million customers.

In 2021:
Archer is played by 22 million customers.
Barbarian is played by 36 million customers.
Cleric is played by 14 million customers.

One year later, in 2022:

- 50% of the people who started with the Archer still play with the Archer, while 30% have switched to Barbarian and 20% have switched to Cleric.
- 60% of the customers who stared with the Barbarian still play with the Barbarian, while 10% have switched to Archer and 30% have switched to Cleric.
- 70% of the customers who stared with the Cleric still play with the Cleric, while 10% have switched to Archer and 20% have switched to Barbarian.

a) Write down the stochastic matrix A which represents the change in each character's popularity from 2021 to 2022, and use it to find the number of people who played with each character in 2022.

b) Suppose the trend continues each year. In the distant future, what will be the most popular character?

You may use the fact that the 1-eigenspace of A is spanned by $\begin{pmatrix} 6 \\ 13 \\ 17 \end{pmatrix}$.

Solution.

a)

$$A = \begin{pmatrix} 0.5 & 0.1 & 0.1 \\ 0.3 & 0.6 & 0.2 \\ 0.2 & 0.3 & 0.7 \end{pmatrix}, \qquad A\begin{pmatrix} 22 \\ 36 \\ 14 \end{pmatrix} = \begin{pmatrix} 16 \\ 31 \\ 25 \end{pmatrix}.$$

This means that, in 2022: the archer, barbarian, and cleric will have 16 million, 31 million, and 25 million players (respectively).

b) Since the 1-eigenspace for the positive stochastic matrix A is spanned by $\begin{pmatrix} 6 \\ 13 \\ 17 \end{pmatrix}$, the steady-state vector for A is

$$\frac{1}{6+13+17}\begin{pmatrix} 6 \\ 13 \\ 17 \end{pmatrix} = \frac{1}{36}\begin{pmatrix} 6 \\ 13 \\ 17 \end{pmatrix} = \begin{pmatrix} 1/6 \\ 13/36 \\ 17/36 \end{pmatrix}.$$

Thus, in the long-term, about 1/6 of the players will use the archer, 13/36 of the players will use the barbarian, and 17/36 of the players will play the cleric. The playerbase is 72 million, so eventually the distribution of players will approximately be the following:

$$\text{Archer}: \frac{1}{6}(72) = 12 \text{ million}$$
$$\text{Barbarian}: \frac{13}{36}(72) = 26 \text{ million}$$
$$\text{Cleric}: \frac{17}{36}(72) = 34 \text{ million.}$$

In the long run, the cleric will be the most popular character.