Math 1553 Worksheet §3.4-3.6

Solutions

1. True or false. Answer true if the statement is *always* true. Otherwise, answer false. If your answer is false, either give an example that shows it is false or (in the case of an incorrect formula) state the correct formula.

a) If A and B are $n \times n$ matrices and both are invertible, then the inverse of AB is $A^{-1}B^{-1}$.

b) If A is an $n \times n$ matrix and the equation $Ax = b$ has at least one solution for each b in $\mathbf{R}^n$, then the solution is *unique* for each b in $\mathbf{R}^n$.

c) If A is an $n \times n$ matrix and the equation $Ax = b$ has at most one solution for each b in $\mathbf{R}^n$, then the solution must be *unique* for each b in $\mathbf{R}^n$.

d) If A and B are invertible $n \times n$ matrices, then $A+B$ is invertible and $(A+B)^{-1} = A^{-1} + B^{-1}$.

e) If A is a 3×4 matrix and B is a 4×2 matrix, then the linear transformation Z defined by $Z(x) = ABx$ has domain $\mathbf{R}^3$ and codomain $\mathbf{R}^2$.

Solution.

a) False. $(AB)^{-1} = B^{-1}A^{-1}$.

b) True. The first part says the transformation $T(x) = Ax$ is onto. Since A is $n \times n$, then it has n pivots. This is the same as saying A is invertible, and there is no free variable. Therefore, the equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^n$.

c) True. The first part says the transformation $T(x) = Ax$ is one-to-one (namely not multiple-to-one). Since A is $n \times n$, then it has n pivots. Then there is no free variable. Therefore, the equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^n$.

d) False. $A + B$ might not be invertible in the first place. For example, if $A = I_2$ and $B = -I_2$ then $A + B = 0$ which is not invertible. Even in the case when $A + B$ is invertible, it still might not be true that $(A+B)^{-1} = A^{-1} + B^{-1}$. For example, $(I_2 + I_2)^{-1} = (2I_2)^{-1} = \frac{1}{2}I_2$, whereas $(I_2)^{-1} + (I_2)^{-1} = I_2 + I_2 = 2I_2$.

e) False. In order for Bx to make sense, x must be in $\mathbf{R}^2$, and so Bx is in $\mathbf{R}^4$ and $A(Bx)$ is in $\mathbf{R}^3$. Therefore, the domain of Z is $\mathbf{R}^2$ and the codomain of Z is $\mathbf{R}^3$.

2. A is $m \times n$ matrix, B is $n \times m$ matrix. Select all correct answers from the box. It is possible to have more than one correct answer.

a) Suppose x is in $\mathbf{R}^m$. Then ABx *must be* in:

$\boxed{\text{Col}(A), \quad \text{Nul}(A), \quad \text{Col}(B), \quad \text{Nul}(B)}$

b) Suppose x in $\mathbf{R}^n$. Then BAx *must be* in:
$\boxed{\text{Col}(A), \quad \text{Nul}(A), \quad \text{Col}(B), \quad \text{Nul}(B)}$

c) If $m > n$, then columns of AB could be linearly $\boxed{independent, \quad dependent}$

d) If $m > n$, then columns of BA could be linearly $\boxed{independent, \quad dependent}$

e) If $m > n$ and $Ax = 0$ has nontrivial solutions, then columns of BA could be linearly $\boxed{independent, \quad dependent}$

Solution.

Recall, AB can be computed as A multiplying every column of B. That is $AB = \begin{pmatrix} Ab_1 & Ab_2 & \cdots Ab_m \end{pmatrix}$ where $B = \begin{pmatrix} b_1 & b_2 & \cdots b_m \end{pmatrix}$.

a) $\boxed{\text{Col}(A)}$. Note Bx is a vector in $\mathbf{R}^n$, so $ABx = A(Bx)$ is multiplying A with a vector in $\mathbf{R}^n$. Therefore, ABx is a linear combination of the columns of A, so ABx must be in Col(A).

b) $\boxed{\text{Col}(B)}$. Similarly, $BAx = B(Ax)$ is multiplying B with a vector in R^m, which is therefore a linear combination of columns of B, so BAx is in Col(B).

c) $\boxed{dependent}$. The fact $m > n$ means A has at most n pivots, so $dim(\text{Col}(A)) \leq n$. From part (a) we know that every vector of the form ABx is in Col(A), which has dimension at most n. This means AB can have at most n pivots. But AB is an $m \times m$ matrix and $m > n$, so AB can't have a pivot in every column and therefore the columns of AB must be linearly dependent.

d) $\boxed{independent, dependent}$. Both are possible. Since $m > n$, we know that A and B have at most n pivots. Here BA is an $n \times n$ matrix, and it is possible (but not guaranteed) for BA to have a pivot in each column. We give two examples below.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{then } BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{then } BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

e) $\boxed{dependent}$. From the second example above, BA has dependent columns, we know "dependent" is one possible answer. Now to see if "independent" is also possible, we need to find out if BA could have n pivots.

Since $Ax = 0$ has nontrivial solution say x^*, then x^* is also a nontrivial solution of $BAx = 0$. That means the equation $BAx = 0$ has at least one free variable, so the columns of BA must be linearly dependent.

To summarize what we are actually study here, there are several relations between these subspaces.
Every vector in Col(AB) is also in Col(A).
Every vector in Col(BA) is also in Col(B).
Every vector in Nul(A) is also in Nul(BA).
Every vector in Nul(B) is also in Nul(AB).

3. Consider the following linear transformations:

$$\begin{aligned} T &: \mathbf{R}^3 \longrightarrow \mathbf{R}^2 && T \text{ projects onto the } xy\text{-plane, forgetting the } z\text{-coordinate} \\ U &: \mathbf{R}^2 \longrightarrow \mathbf{R}^2 && U \text{ rotates clockwise by } 90^\circ \\ V &: \mathbf{R}^2 \longrightarrow \mathbf{R}^2 && V \text{ scales the } x\text{-direction by a factor of 2.} \end{aligned}$$

Let A, B, C be the matrices for T, U, V, respectively.

a) Write A, B, and C.

b) Compute the matrix for $V \circ U \circ T$.

c) Compute the matrix for $U \circ V \circ T$.

d) Describe U^{-1} and V^{-1}, and compute their matrices.

Solution.

a) We plug in the unit coordinate vectors:

$$\begin{array}{llllll} T(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & T(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} & T(e_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \Longrightarrow & A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ & U(e_1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} & U(e_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \Longrightarrow & B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & . \\ & V(e_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} & V(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \Longrightarrow & C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

b) $CBA = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}$.

c) $BCA = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}$.

d) Intuitively, if we wish to "undo" U, we can imagine that $\begin{pmatrix} x \\ y \end{pmatrix}$. To do this, we need to rotate it 90° *counterclockwise*. Therefore, U^{-1} is counterclockwise rotation by 90°.

Similarly, to undo the transformation V that scales the x-direction by 2, we need to scale the x-direction by $1/2$, so V^{-1} scales the x-direction by a factor of $1/2$.

Their matrices are, respectively,

$$B^{-1} = \frac{1}{0 \cdot 0 - (-1) \cdot 1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$C^{-1} = \frac{1}{2 \cdot 1 - 0 \cdot 0} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}.$$