Math 1553 Worksheet §5.2 - §5.4

Solutions

1. Suppose A is an $n \times n$ matrix satisfying $A^2 = 0$. Find all eigenvalues of A. Justify your answer.

Solution.

If λ is an eigenvalue of A and $v \neq 0$ is a corresponding eigenvector, then

$$Av = \lambda v \implies A(Av) = A\lambda v \implies A^2 v = \lambda(Av) \implies 0 = \lambda(\lambda v) \implies 0 = \lambda^2 v.$$

Since $v \neq 0$ this means $\lambda^2 = 0$, so $\lambda = 0$. This shows that 0 is the only possible eigenvalue of A.

On the other hand, $\det(A) = 0$ since $(\det(A))^2 = \det(A^2) = \det(0) = 0$, so 0 must be an eigenvalue of A. Therefore, the only eigenvalue of A is 0.

2. Answer yes, no, or maybe. Justify your answers. In each case, A is a matrix whose entries are real numbers.

a) Suppose $A = \begin{pmatrix} 3 & 0 & 0 \\ 5 & 1 & 0 \\ -10 & 4 & 7 \end{pmatrix}$. Then the characteristic polynomial of A is

$$\det(A - \lambda I) = (3-\lambda)(1-\lambda)(7-\lambda).$$

b) If A is a 3×3 matrix with characteristic polynomial $-\lambda(\lambda-5)^2$, then the 5-eigenspace is 2-dimensional.

c) If A is an invertible 2×2 matrix, then A is diagonalizable.

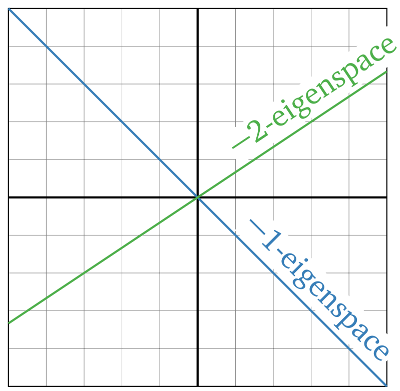
Solution.

a) Yes. Since $A - \lambda I$ is triangular, its determinant is the product of its diagonal entries.

b) Maybe. The geometric multiplicity of $\lambda = 5$ can be 1 or 2. For example, the matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 2-dimensional, whereas the matrix $\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 1-dimensional. Both matrices have characteristic polynomial $-\lambda(5-\lambda)^2$.

c) Maybe. The identity matrix is invertible and diagonalizable, but the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible but not diagonalizable.

3. The eigenspaces of some 2×2 matrix A are drawn below. Write an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$. Can you find another pair of C and D so that $A = CDC^{-1}$?



Solution.

We choose D to be a diagonal matrix whose entries are the eigenvalues of A, and C a matrix whose columns are corresponding eigenvectors (written in the same order).

The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -2$.
The (-1)-eigenspace is spanned by $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The (-2)-eigenspace is spanned by $v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Therefore, we can choose $C = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$.

There are many possibilities for C and D.
For example, since $\text{Span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} = \text{Span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$, we could have chosen $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ instead, so that

$$C = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}.$$

Alternatively, we could have rearranged the order of the diagonal entries of D and taken care to use the corresponding order in the columns of C:

$$C = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Regardless, if you write any correct answers for C and D and go the extra step of carrying out the computation, you will obtain

$$A = CDC^{-1} = -\frac{1}{5}\begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}.$$

4. Suppose A is a 2×2 matrix satisfying

$$A\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \qquad A\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

a) Diagonalize A by finding 2×2 matrices C and D (with D diagonal) so that $A = CDC^{-1}$.

b) Find A^{17}.

Solution.

a) From the information given, $\lambda_1 = -2$ is an eigenvalue for A with corresponding eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and $\lambda_2 = 0$ is an eigenvalue with eigenvector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

By the Diagonalization Theorem, $A = CDC^{-1}$ where

$$C = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix}, \qquad D = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}.$$

b) We find $C^{-1} = \dfrac{1}{-3+2}\begin{pmatrix} 3 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix}$.

$$\begin{aligned} A^{17} = CD^{17}C^{-1} &= \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} (-2)^{17} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 \cdot 2^{17} & 2^{18} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -3 \cdot 2^{17} & -2^{18} \\ 3 \cdot 2^{17} & 2^{18} \end{pmatrix}. \end{aligned}$$