Section 2.6

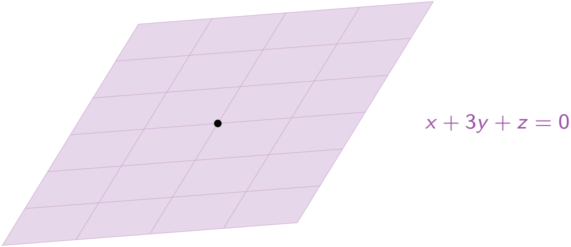
Subspaces

Motivation

Today we will discuss **subspaces** of $\mathbf{R}^n$.

A subspace turns out to be the same as a span, except we don't know *which* vectors it's the span of.

This arises naturally when you have, say, a plane through the origin in $\mathbf{R}^3$ which is *not* defined (a priori) as a span, but you still want to say something about it.



Definition of Subspace

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Fast-forward

Every subspace is a span, and every span is a subspace.

A subspace is a span of some vectors, but you haven't computed what those vectors are yet.

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What does this mean?

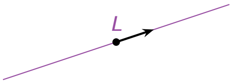
- If v is in V, then all scalar multiples of v are in V by (3). In other words, the line through any nonzero vector in V is also in V.
- If u, v are in V, then cu and dv are in V for any scalars c, d by (3). So $cu + dv$ is in V by (2). So $\text{Span}\{u, v\}$ is contained in V.
- Likewise, if $v_1, v_2, \ldots, v_n$ are all in V, then $\text{Span}\{v_1, v_2, \ldots, v_n\}$ is contained in V: a subspace contains the span of any set of vectors in it.

If you pick enough vectors in V, eventually their span will fill up V, so:

> A subspace is a span of some set of vectors in it.

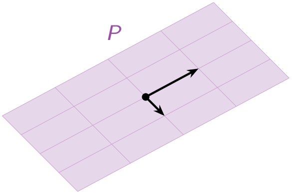
Example

A line L through the origin is a subspace: L contains zero and is easily seen to be closed under addition and scalar multiplication.



Example

A plane P through the origin is a subspace: P contains zero; the sum of two vectors in P is also in P; and any scalar multiple of a vector in P is also in P.



Example

All of $\mathbf{R}^n$: this contains 0, and is closed under addition and scalar multiplication.

Example

The subset $\{0\}$: this subspace contains only one vector.

Note these are all pictures of spans! (Line, plane, space, etc.)

Subsets and Subspaces

They aren't the same thing

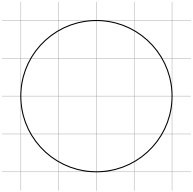
A **subset** of $\mathbf{R}^n$ is any collection of vectors in $\mathbf{R}^n$ whatsoever. For example, the unit circle

$$C = \{(x, y) \text{ in } \mathbf{R}^2 \mid x^2 + y^2 = 1\}$$

is a subset of $\mathbf{R}^2$, but it is not a subspace.

All of the following non-examples on the next slide are still subsets.

A **subspace** is a special kind of subset, that satisfies the three defining properties.

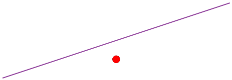


Subset: *yes*
Subspace: *no*

Non-Examples

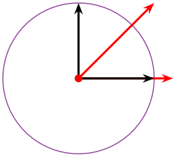
Non-Example

A line L (or any other set) that doesn't contain the origin is not a subspace. Fails: 1.



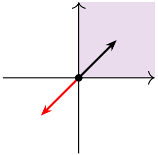
Non-Example

A circle C is not a subspace. Fails: 1,2,3. Think: a circle isn't a "linear space."



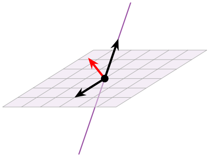
Non-Example

The first quadrant in $\mathbf{R}^2$ is not a subspace. Fails: 3 only.



Non-Example

A line union a plane in $\mathbf{R}^3$ is not a subspace. Fails: 2 only.



Subspaces are Spans, and Spans are Subspaces

Theorem

Any $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is a subspace.

!!!

Every subspace is a span, and every span is a subspace.

Definition

If $V = \text{Span}\{v_1, v_2, \ldots, v_p\}$, we say that V is the subspace **generated by** or **spanned by** the vectors $v_1, v_2, \ldots, v_p$. We call $\{v_1, v_2, \ldots, v_p\}$ a **spanning set** for V.

Check:

1. $0 = 0v_1 + 0v_2 + \cdots + 0v_p$ is in the span.
2. If, say, $u = 3v_1 + 4v_2$ and $v = -v_1 - 2v_2$, then

$$u + v = 3v_1 + 4v_2 - v_1 - 2v_2 = 2v_1 + 2v_2$$

   is also in the span.

3. Similarly, if u is in the span, then so is cu for any scalar c.

Poll

Poll

Which of the following are subspaces?

A. The empty set $\{\}$.

B. The solution set to a homogeneous system of linear equations.

C. The solution set to an inhomogeneous system of linear equations.

D. The set of all vectors in $\mathbf{R}^n$ with rational (fraction) coordinates.

For the ones which are not subspaces, which property(ies) do they not satisfy?

A. This is not a subspace: it does not contain the zero vector.

B. This is a subspace: the solution set is a span, produced by finding the parametric vector form of the solution.

C. This is not a subspace: it does not contain 0.

D. This is not a subspace: it is not closed under multiplication by scalars (e.g. by π).

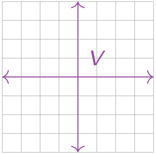
Subspaces
Verification

Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; ab = 0 \right\}$. Let's check if V is a subspace or not.

1. Does V contain the zero vector? $\binom{a}{b} = \binom{0}{0} \implies ab = 0$

3. Is V closed under scalar multiplication?
   - Let $\binom{a}{b}$ be (an unknown vector) in V.
   - *This means:* a and b are numbers such that $ab = 0$.
   - Let c be a scalar. Is $c\binom{a}{b} = \binom{ca}{cb}$ in V?
   - *This means:* $(ca)(cb) = 0$.
   - Well, $(ca)(cb) = c^2(ab) = c^2(0) = 0$

2. Is V closed under addition?
   - Let $\binom{a}{b}$ and $\binom{a'}{b'}$ be (unknown vectors) in V.
   - *This means:* $ab = 0$ and $a'b' = 0$.
   - Is $\binom{a}{b} + \binom{a'}{b'} = \binom{a+a'}{b+b'}$ in V?
   - *This means:* $(a + a')(b + b') = 0$.
   - This is not true for all such a, a', b, b': for instance, $\binom{1}{0}$ and $\binom{0}{1}$ are in V, but their sum $\binom{1}{0} + \binom{0}{1} = \binom{1}{1}$ is not in V, because $1 \cdot 1 \neq 0$.



We conclude that V is *not* a subspace. A picture is above. (It doesn't look like a span.)

Column Space and Null Space

An $m \times n$ matrix A naturally gives rise to *two* subspaces.

Definition

- The **column space** of A is the subspace of $\mathbf{R}^m$ spanned by the columns of A. It is written $\operatorname{Col} A$.
- The **null space** of A is the set of all solutions of the homogeneous equation $Ax = 0$:
$$\operatorname{Nul} A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$
This is a subspace of $\mathbf{R}^n$.

The column space is defined as a span, so we know it is a subspace.

Check that the null space is a subspace:

1. 0 is in $\operatorname{Nul} A$ because $A0 = 0$.
2. If u and v are in $\operatorname{Nul} A$, then $Au = 0$ and $Av = 0$. Hence
$$A(u + v) = Au + Av = 0,$$
so $u + v$ is in $\operatorname{Nul} A$.
3. If u is in $\operatorname{Nul} A$, then $Au = 0$. For any scalar c, $A(cu) = cAu = 0$. So cu is in $\operatorname{Nul} A$.

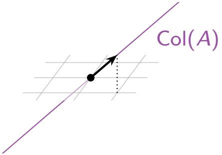
Column Space and Null Space

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Let's compute the column space:

$$\operatorname{Col} A = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$



This is a line in $\mathbf{R}^3$.

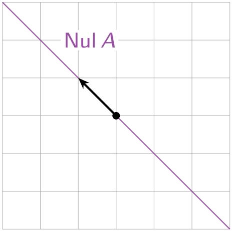
Let's compute the null space:

The reduced row echelon form of A is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

This gives the equation $x + y = 0$, or

$$\begin{aligned} x &= -y \\ y &= y \end{aligned} \quad \xrightarrow{\text{parametric vector form}} \quad \begin{pmatrix} x \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence the null space is $\operatorname{Span}\{\binom{-1}{1}\}$, a line in $\mathbf{R}^2$.



The Null Space is a Span

The column space of a matrix A is defined to be a span (of the columns).

The null space is defined to be the solution set to $Ax = 0$. It is a subspace, so it is a span.

Question

How to find vectors that span the null space?

Answer: Parametric vector form! We know that the solution set to $Ax = 0$ has a parametric form that looks like

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

if, say, x_3 and x_4 are the free variables. So

$$\operatorname{Nul} A = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Refer back to the slides for §2.4 (Solution Sets).

Note: It is much easier to define the null space first as a subspace, then find spanning vectors *later*, if we need them. This is one reason subspaces are so useful.

- A **subspace** is the same as a span of some number of vectors, but we haven't computed the vectors yet.
- To any matrix is associated two subspaces, the **column space** and the **null space**:

$$\begin{aligned}\operatorname{Col} A &= \text{the span of the columns of } A \\ \operatorname{Nul} A &= \text{the solution set of } Ax = 0.\end{aligned}$$

How do you check if a subset is a subspace?

- Is it a span? Can it be written as a span?
- Can it be written as the column space of a matrix?
- Can it be written as the null space of a matrix?
- Is it all of $\mathbf{R}^n$ or the zero subspace $\{0\}$?
- Can it be written as a type of subspace that we'll learn about later (eigenspaces, $\ldots$)?

If so, then it's automatically a subspace.

If all else fails:

- Can you verify directly that it satisfies the three defining properties?