Supplemental problems: §5.1

1. True or false. Answer true if the statement is always true. Otherwise, answer false.

a) If A and B are $n \times n$ matrices and A is row equivalent to B, then A and B have the same eigenvalues.

b) If A is an $n \times n$ matrix and its eigenvectors form a basis for $\mathbf{R}^n$, then A is invertible.

c) If 0 is an eigenvalue of the $n \times n$ matrix A, then $\operatorname{rank}(A) < n$.

d) The diagonal entries of an $n \times n$ matrix A are its eigenvalues.

e) If A is invertible and 2 is an eigenvalue of A, then $\frac{1}{2}$ is an eigenvalue of A^{-1}.

f) If $\det(A) = 0$, then 0 is an eigenvalue of A.

Solution.

a) False. For instance, the matrices $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are row equivalent, but have different eigenvalues.

b) False. For example, the zero matrix is not invertible but its eigenvectors form a basis for $\mathbf{R}^n$.

c) True. If $\lambda = 0$ is an eigenvalue of A then A is not invertible so its associated transformation $T(x) = Ax$ is not onto, hence $\operatorname{rank}(A) < n$.

d) False. This is true if A is triangular, but not in general.

For example, if $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ then the diagonal entries are 2 and 0 but the only eigenvalue is $\lambda = 1$, since solving the characteristic equation gives us

$$(2-\lambda)(-\lambda) - (1)(-1) = 0 \qquad \lambda^2 - 2\lambda + 1 = 0 \qquad (\lambda - 1)^2 = 0 \qquad \lambda = 1.$$

e) True. Let v be an eigenvector corresponding to the eigenvalue 2.

$$Av = 2v \implies A^{-1}Av = A^{-1}(2v) \implies v = 2A^{-1}v \implies \frac{1}{2}v = A^{-1}v.$$

Therefore, v is an eigenvector of A^{-1} corresponding to the eigenvalue $\frac{1}{2}$.

f) True. If $\det(A) = 0$ then A is not invertible, so $Av = 0v$ has a nontrivial solution.

2. In this problem, you need not explain your answers; just circle the correct one(s).

Let A be an $n \times n$ matrix.

a) Which **one** of the following statements is correct?

1. An eigenvector of A is a vector v such that $Av = \lambda v$ for a nonzero scalar λ.

2. An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a scalar λ.
3. An eigenvector of A is a nonzero scalar λ such that $Av = \lambda v$ for some vector v.
4. An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a nonzero scalar λ.

b) Which **one** of the following statements is **not** correct?

1. An eigenvalue of A is a scalar λ such that $A - \lambda I$ is not invertible.
2. An eigenvalue of A is a scalar λ such that $(A - \lambda I)v = 0$ has a solution.
3. An eigenvalue of A is a scalar λ such that $Av = \lambda v$ for a nonzero vector v.
4. An eigenvalue of A is a scalar λ such that $\det(A - \lambda I) = 0$.

Solution.

a) Statement 2 is correct: an eigenvector must be nonzero, but its eigenvalue may be zero.

b) Statement 2 is incorrect: the solution v must be nontrivial.

3. Find a basis $\mathcal{B}$ for the (-1)-eigenspace of $Z = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$

Solution.

For $\lambda = -1$, we find $\mathrm{Nul}(Z - \lambda I)$.

$$\left(Z - \lambda I \mid 0 \right) = \left(Z + I \mid 0 \right) = \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, $x = -y$, $y = y$, and $z = 0$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

A basis is $\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$. We can check to ensure $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector with corresponding eigenvalue -1:

$$Z \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 + 3 \\ -3 + 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

4. Suppose A is an $n \times n$ matrix satisfying $A^2 = 0$. Find all eigenvalues of A. Justify your answer.

Solution.

If λ is an eigenvalue of A and $v \neq 0$ is a corresponding eigenvector, then

$$Av = \lambda v \implies A(Av) = A\lambda v \implies A^2 v = \lambda(Av) \implies 0 = \lambda(\lambda v) \implies 0 = \lambda^2 v.$$

Since $v \neq 0$ this means $\lambda^2 = 0$, so $\lambda = 0$. This shows that 0 is the only possible eigenvalue of A.

On the other hand, $\det(A) = 0$ since $(\det(A))^2 = \det(A^2) = \det(0) = 0$, so 0 must be an eigenvalue of A. Therefore, the only eigenvalue of A is 0.

5. Match the statements (i)-(v) with the corresponding statements (a)-(e). All matrices are 3×3. There is unique correspondence. Justify the correspondences in words.

(i) $Ax = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ has a unique solution.
(ii) The transformation $T(v) = Av$ fixes a nonzero vector.
(iii) A is obtained from B by subtracting the third row of B from the first row of B.
(iv) The columns of A and B are the same; except that the first, second and third columns of A are respectively the first, third, and second columns of B.
(v) The columns of A, when added, give the zero vector.

(a) 0 is an eigenvalue of A.
(b) A is invertible.
(c) $\det(A) = \det(B)$
(d) $\det(A) = -\det(B)$
(e) 1 is an eigenvalue of A.

Solution.

(i) matches with (b).
(ii) matches with (e).
(iii) matches with (c).
(iv) matches with (d).
(v) matches with (a).

Supplemental problems: §5.2

1. True or false. If the statement is always true, answer true and justify why it is true. Otherwise, answer false and give an example that shows it is false.

a) If A and B are $n \times n$ matrices with the same eigenvectors, then A and B have the same characteristic polynomial.

b) If A is a 3×3 matrix with characteristic polynomial $-\lambda^3 + \lambda^2 + \lambda$, then A is invertible.

Solution.

a) False: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ have the same eigenvectors (all nonzero vectors in $\mathbf{R}^2$) but characteristic polynomials λ^2 and $(1-\lambda)^2$, respectively.

b) False: $\lambda = 0$ is a root of the characteristic polynomial, so 0 is an eigenvalue, and A is not invertible.

2. Find all values of a so that $\lambda = 1$ an eigenvalue of the matrix A below.

$$A = \begin{pmatrix} 3 & -1 & 0 & a \\ a & 2 & 0 & 4 \\ 2 & 0 & 1 & -2 \\ 13 & a & -2 & -7 \end{pmatrix}$$

Solution.

We need to know which values of a make the matrix $A - I_4$ noninvertible. We have

$$A - I_4 = \begin{pmatrix} 2 & -1 & 0 & a \\ a & 1 & 0 & 4 \\ 2 & 0 & 0 & -2 \\ 13 & a & -2 & -8 \end{pmatrix}.$$

We expand cofactors along the third column, then the second column:

$$\begin{aligned} \det(A - I_4) &= 2\det\begin{pmatrix} 2 & -1 & a \\ a & 1 & 4 \\ 2 & 0 & -2 \end{pmatrix} \\ &= (2)(1)\det\begin{pmatrix} a & 4 \\ 2 & -2 \end{pmatrix} + (2)(1)\det\begin{pmatrix} 2 & a \\ 2 & -2 \end{pmatrix} \\ &= 2(-2a-8) + 2(-4-2a) = -8a - 24. \end{aligned}$$

This is zero if and only if $a = -3$.