MATH 1553, FALL 2023
SAMPLE MIDTERM 2A: COVERS 2.5 - 3.4

Name		**GT ID**	

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and all work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. We recommend completing the practice exam in 75 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to section §2.5 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to §§2.5 through 3.4.

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Problem 1.

For each statement, answer TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

a) Suppose v_1, v_2, and v_3 are linearly dependent vectors in $\mathbf{R}^4$. Then v_1 must be a linear combination of v_2 and v_3.

TRUE FALSE

b) If A is a 3×8 matrix, then $\dim(\mathrm{Nul}\, A) > \dim(\mathrm{Col}\, A)$.

TRUE FALSE

c) Consider the subspace W of $\mathbf{R}^4$ given by

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x - y - z + w = 0 \right\}.$$

Then $\dim(W) = 3$.

TRUE FALSE

d) Suppose $T : \mathbf{R}^4 \to \mathbf{R}^3$ is a transformation. Then for each y in $\mathbf{R}^3$, there is a vector x in $\mathbf{R}^4$ so that $T(x) = y$.

TRUE FALSE

e) There is a 3×4 matrix whose null space is a plane and whose column space is a line.

TRUE FALSE

Problem 2.

Parts (a), (b), and (c) are unrelated. There is no work required and no partial credit on this page.

a) (4 points) In each case, clearly circle YES or NO.

(i) Let V be the set of all vectors of the form $\begin{pmatrix} x \\ 0 \end{pmatrix}$ in $\mathbf{R}^2$, where x is any real number.

Is V a subspace of $\mathbf{R}^2$? YES NO

(ii) Let W be the set in $\mathbf{R}^3$ consisting of all solutions to the vector equation

$$x_1 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Is W a subspace of $\mathbf{R}^3$? YES NO

(iii) Suppose A is a 3×3 matrix. Must it be true that the solution set of the matrix equation $Ax = 0$ is a subspace of $\mathbf{R}^3$? YES NO

(iv) Suppose $T : \mathbf{R}^2 \to \mathbf{R}^4$ is a linear transformation. Must it be true that the range of T is a subspace of $\mathbf{R}^4$? YES NO

b) (3 points) Suppose $\{v_1, v_2, v_3, v_4\}$ is a **linearly independent** set of vectors in $\mathbf{R}^4$. Which of the following statements are true? Clearly circle all that apply.

(i) For each b in $\mathbf{R}^4$, the vector equation

$$x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 = b$$

is consistent and has a unique solution.

(ii) It is possible that the set $\{v_1, v_2, v_3\}$ is linearly dependent.

(iii) $\mathrm{Span}\{v_1, v_2, v_3, v_4\} = \mathbf{R}^4$.

c) (3 points) Suppose $\{u, v, w\}$ is a basis for some subspace V of $\mathbf{R}^n$. Which of the following must be true? Clearly circle all that apply.

(i) If $\{a, b, c\}$ are vectors in V and $\mathrm{Span}\{a, b, c\} = V$, then $\{a, b, c\}$ must be a basis for V.

(ii) The set $\{u,\ u + 2v,\ v + w\}$ must be a basis for V.

(iii) If $\{a, b, c\}$ is any set of 3 linearly independent vectors in V, then $\{a, b, c\}$ must be a basis for V.

Problem 3.

Parts (a), (b), and (c) are unrelated. You do not need to show your work, and there is no partial credit.

a) (3 points) Consider the set $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; x = y \right\}$.

(i) Is V closed under addition? YES NO

(ii) Is V closed under scalar multiplication? YES NO

(iii) Is there a matrix A so that $\text{Col}(A) = V$? YES NO

b) (4 points) Suppose A is a 3×4 matrix and B is a 4×5 matrix, and let T be the matrix transformation $T(x) = ABx$. Which of the following must be true? Clearly circle all that apply.

(i) The null space of AB is a subspace of $\mathbf{R}^4$.

(ii) Every vector in the column space of AB is also in the column space of A.

(iii) T cannot be one-to-one.

c) (3 points) Which of the following transformations are linear? Clearly circle all that apply.

(i) $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x_1, x_2) = (x_1, |x_2|)$.

(ii) $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by $T(x_1, x_2, x_3) = (x_1 - x_2, x_3, x_1)$.

(iii) $T : \mathbf{R}^2 \to \mathbf{R}^2$ that reflects vectors across the line $y = -x$.

Problem 4.

You do not need to show your work on this problem, and there is no partial credit. Parts (a), (b), and (c) are unrelated.

a) (3 points) In each case, consider the matrix transformation $T(x) = Ax$. Determine whether T is one-to-one and whether T is onto. If T is one-to-one, clearly circle "one-to-one." If T is onto, clearly circle "onto." If T is neither one-to-one nor onto, do not circle anything. If T is one-to-one and onto, circle one-to-one and circle onto.

(I) $A = \begin{pmatrix} \cos(\pi/10) & -\sin(\pi/10) \\ \sin(\pi/10) & \cos(\pi/10) \end{pmatrix}$ one-to-one onto

(II) $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 2 \end{pmatrix}$ one-to-one onto

(III) $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ one-to-one onto

b) (4 points) Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix}$, and let T be the matrix transformation $T(x) = Ax$.

(I) What is the domain of T? Clearly circle your answer below.

$\mathbf{R}$ $\mathbf{R}^2$ $\mathbf{R}^3$ $\mathbf{R}^4$ $\mathbf{R}^5$

(II) What is the codomain of T? Clearly circle your answer below.

$\mathbf{R}$ $\mathbf{R}^2$ $\mathbf{R}^3$ $\mathbf{R}^4$ $\mathbf{R}^5$

(III) What is the null space of A? Clearly circle your answer below.

a point in $\mathbf{R}^2$ a line in $\mathbf{R}^2$ a point in $\mathbf{R}^3$ a line in $\mathbf{R}^3$ a plane in $\mathbf{R}^3$

(IV) Is T onto? Clearly circle your answer below.

YES NO

c) (3 points) Consider the transformation

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

(i) Write a matrix A so that $T(x) = Ax$.

(ii) Is there a set of three linearly independent vectors in the range of T?

Problem 5.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

Consider the matrix A and its reduced row echelon form given below.

$$A = \begin{pmatrix} -4 & 4 & -8 & -13 \\ 3 & -3 & 6 & 10 \\ -5 & 5 & -10 & -16 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) (2 points) Write a basis for Col A. There is no work required on this part.

b) (4 points) Find a basis for Nul A.

c) (2 points) Write one nonzero vector in the null space of A. There is no work required and no partial credit for this part.

d) (2 pts) Let T be the matrix transformation $T(x) = Ax$. Are there two different vectors u and v (with $u \neq v$) satisfying $T(u) = T(v)$?
If your answer is yes, write such vectors u and v. If your answer is no, justify why not.

Problem 6.

Free response. Show your work except in part (c). A correct answer without sufficient work may receive little or no credit. In this problem:

$T : \mathbf{R}^3 \to \mathbf{R}^2$ is the linear transformation $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y + 2z \\ z - x \end{pmatrix}$.

$U : \mathbf{R}^2 \to \mathbf{R}^2$ is the linear transformation that rotates vectors **clockwise** by 45 degrees.
$V : \mathbf{R}^2 \to \mathbf{R}^2$ is the linear transformation that reflects vectors across the line $y = x$.

a) (3 points) Find the standard matrix A for T.

b) (2 points) Write the standard matrix B for U.
(do *not* leave your answer in terms of sine and cosine; simplify it completely)

c) (2 points) Write the standard matrix C for V.

d) (3 pts) Let $W : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that first reflects vectors across the line $y = x$, then rotates by 45 degrees clockwise. Find the standard matrix D for W.

Problem 7.

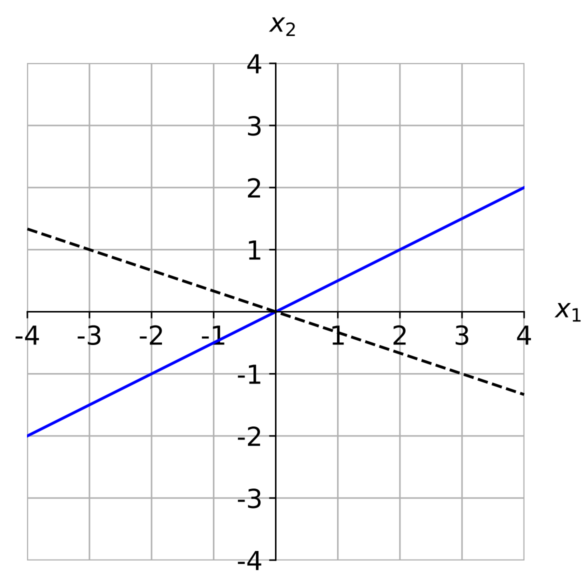
Free response. Show your work! A correct answer without sufficient work may receive little or no credit. Parts (a), (b), and (c) are unrelated.

a) (3 points) Suppose $T : \mathbf{R}^3 \to \mathbf{R}^2$ is a linear transformation satisfying

$$T\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Find $T\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$.

b) (4 points) Find a matrix A whose **column space** is the **dotted** line below and whose null space is the solid diagonal line below.



c) (3 points) Let A be the matrix that reflects vectors counterclockwise by 10 degrees. Find $A^9\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.