Math 1553 Worksheet §5.4 - §5.5

Solutions

1. Write a matrix that is invertible but not diagonalizable.

Solution.

The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible but not diagonalizable.

2. Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Find all eigenvalues of A. For each eigenvalue, find an associated eigenvector.

Solution.

The characteristic polynomial is

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 5$$

$$\lambda^2 - 2\lambda + 5 = 0 \iff \lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

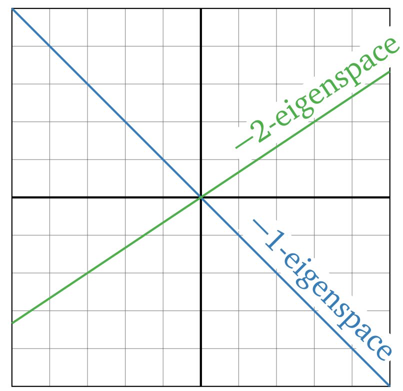
For the eigenvalue $\lambda = 1 - 2i$, we use the shortcut trick you may have seen in class: the first row $\begin{pmatrix} a & b \end{pmatrix}$ of $A - \lambda I$ will lead to an eigenvector $\begin{pmatrix} -b \\ a \end{pmatrix}$ (or equivalently, $\begin{pmatrix} b \\ -a \end{pmatrix}$ if you prefer).

$$\begin{pmatrix} A - (1-2i)I \mid 0 \end{pmatrix} = \left(\begin{array}{cc|c} 2i & 2 & 0 \\ (*) & (*) & 0 \end{array}\right) \qquad \Longrightarrow \qquad v = \begin{pmatrix} -2 \\ 2i \end{pmatrix}.$$

From the correspondence between conjugate eigenvalues and their eigenvectors, we know (without doing any additional work!) that for the eigenvalue $\lambda = 1 + 2i$, a corresponding eigenvector is $w = \overline{v} = \begin{pmatrix} -2 \\ -2i \end{pmatrix}$.

If you used row-reduction for finding eigenvectors, you would find $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$ as an eigenvector for eigenvalue $1 - 2i$, and $w = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ as an eigenvector for eigenvalue $1 + 2i$.

3. The eigenspaces of some 2×2 matrix A are drawn below. Write an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$. Can you find another pair of C and D so that $A = CDC^{-1}$?



Solution.

We choose D to be a diagonal matrix whose entries are the eigenvalues of A, and C a matrix whose columns are corresponding eigenvectors (written in the same order).

The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = -2$.
The (-1)-eigenspace is spanned by $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The (-2)-eigenspace is spanned by $v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Therefore, we can choose $C = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$.

There are many possibilities for C and D.
For example, since $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} = \mathrm{Span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$, we could have chosen

$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ instead, so that

$$C = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}.$$

Alternatively, we could have rearranged the order of the diagonal entries of D and taken care to use the corresponding order in the columns of C:

$$C = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

Regardless, if you write any correct answers for C and D and go the extra step of carrying out the computation, you will obtain

$$A = CDC^{-1} = -\frac{1}{5}\begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}.$$

4. Suppose A is a 2×2 matrix satisfying

$$A\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \qquad A\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

a) Diagonalize A by finding 2×2 matrices C and D (with D diagonal) so that $A = CDC^{-1}$.

b) Find A^{17}.

Solution.

a) From the information given, $\lambda_1 = -2$ is an eigenvalue for A with corresponding eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and $\lambda_2 = 0$ is an eigenvalue with eigenvector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

By the Diagonalization Theorem, $A = CDC^{-1}$ where

$$C = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix}, \qquad D = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}.$$

b) We find $C^{-1} = \dfrac{1}{-3+2}\begin{pmatrix} 3 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix}$.

$$\begin{aligned} A^{17} = CD^{17}C^{-1} &= \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} (-2)^{17} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 \cdot 2^{17} & 2^{18} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -3 \cdot 2^{17} & -2^{18} \\ 3 \cdot 2^{17} & 2^{18} \end{pmatrix}. \end{aligned}$$