Math 1553 Worksheet §5.6 - §6.5

Solutions

1. Courage Soda and Dexter Soda compete for a market of 210 customers who drink soda each day.
Today, Courage has 80 customers and Dexter has 130 customers. Each day:

70% of Courage Soda's customers keep drinking Courage Soda, while 30% switch to Dexter Soda.

40% of Dexter Soda's customers keep drinking Dexter Soda, while 60% switch to Courage Soda.

a) Write a stochastic matrix A and a vector x so that Ax will give the number of customers for Courage Soda and Dexter Soda (in that order) tomorrow.
You do not need to compute Ax.

$$A = \begin{pmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{pmatrix} \text{ and } x = \begin{pmatrix} 80 \\ 130 \end{pmatrix}.$$

b) By finding the 1-eigenspace, work shows that the steady state vector is

$$w = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}.$$

Using this determine the following: in the long run, roughly how many daily customers will Courage Soda have?

As n gets large, $A^n \begin{pmatrix} 80 \\ 130 \end{pmatrix}$ approaches $210 \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 140 \\ 70 \end{pmatrix}$. Courage will have roughly 140 customers.

2. **a)** Find the standard matrix B for proj_W, where $W = \text{Span}\left\{\begin{pmatrix}1\\1\\-1\end{pmatrix}\right\}$.

b) What are the eigenvalues of B? Is B is diagonalizable?

c) Let $x = \begin{pmatrix}3\\0\\9\end{pmatrix}$. Find the orthogonal decomposition of x with respect to W.

In other words, find x_W in W and $x_{W^\perp}$ in $W^\perp$ so that $x = x_W + x_{W^\perp}$.

Solution.

a) We use the formula $B = \dfrac{1}{u\cdot u}uu^T$ where $u = \begin{pmatrix}1\\1\\-1\end{pmatrix}$ (this is the formula $B = A(A^TA)^{-1}A^T$ when "A" is just the single vector u).

$$B = \frac{1}{1(1)+1(1)+(-1)(-1)}\begin{pmatrix}1\\1\\-1\end{pmatrix}\begin{pmatrix}1 & 1 & -1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1 & 1 & -1\\ 1 & 1 & -1\\ -1 & -1 & 1\end{pmatrix}$$

$$\implies B = \frac{1}{3}\begin{pmatrix}1 & 1 & -1\\ 1 & 1 & -1\\ -1 & -1 & 1\end{pmatrix}.$$

b) $Bx = x$ for every x in W, and $Bx = 0$ for every x in $W^\perp$, so B has two eigenvalues: $\lambda_1 = 1$ with algebraic and geometric multiplicity 1, $\lambda_2 = 0$ with algebraic and geometric multiplicity 2. Therefore, B is diagonalizable. In fact, if we wanted to, we actually could have actually computed B through diagonalization! Here $v_1 = \begin{pmatrix}1\\1\\-1\end{pmatrix}$ is an eigenvector for $\lambda_1 = 1$, whereas $v_2 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$ and $v_3 = \begin{pmatrix}1\\0\\1\end{pmatrix}$ are linearly independent vectors that are orthogonal to v_1, so they span the eigenspace for $\lambda_2 = 0$. Therefore

$$B = \begin{pmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ -1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\ 1 & -1 & 0\\ -1 & 0 & 1\end{pmatrix}^{-1}$$

c) It follows that

$$x_W = \frac{1}{3}\begin{pmatrix}1 & 1 & -1\\ 1 & 1 & -1\\ -1 & -1 & 1\end{pmatrix}\begin{pmatrix}3\\0\\9\end{pmatrix} = \begin{pmatrix}-2\\-2\\2\end{pmatrix}.$$

Hence, as

$$x_{W^\perp} = x - x_W,$$

we have that

$$x_{W^\perp} = \begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}.$$

Thus, $x = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$.

3. Use least-squares to find the best fit line $y = Ax + B$ through the points $(0,0)$, $(1,8)$, $(3,8)$, and $(4,20)$.

Solution.

We want to find a least squares solution to the system of linear equations

$$\begin{aligned} 0 &= A(0) + B \\ 8 &= A(1) + B \\ 8 &= A(3) + B \\ 20 &= A(4) + B \end{aligned} \qquad \iff \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}.$$

We compute

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 26 & 8 \\ 8 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 112 \\ 36 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 26 & 8 & 112 \\ 8 & 4 & 36 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right).$$

Hence the least squares solution is $A = 4$ and $B = 1$, so the best fit line is $y = 4x + 1$.