Math 1553 Worksheet §3.4

Solutions

1. True or false. Answer true if the statement is *always* true. Otherwise, answer false. If your answer is false, either give an example that shows it is false or (in the case of an incorrect formula) state the correct formula.

a) If A is an $n \times n$ matrix and the equation $Ax = b$ has at least one solution for each b in $\mathbf{R}^n$, then the solution is *unique* for each b in $\mathbf{R}^n$.

b) If A is a 3×4 matrix and B is a 4×2 matrix, then the linear transformation Z defined by $Z(x) = ABx$ has domain $\mathbf{R}^3$ and codomain $\mathbf{R}^2$.

c) Suppose A and B are matrices so that the product AB is defined, and suppose that the transformations $T(v) = Av$ and $U(x) = Bx$ are one-to-one. Then the transformation $T \circ U$ must also be one-to-one.

Solution.

a) True. The first part says the transformation $T(x) = Ax$ is onto, so A has a pivot in every row and therefore has n pivots. Since A is $n \times n$, this means A also has a pivot in every column, so we will never get a free variable in a solution set. Therefore, for each b in $\mathbf{R}^n$, the equation $Ax = b$ will be consistent and have exactly one solution.

b) False. The matrix AB is a 3×2 matrix, so the domain of Z is $\mathbf{R}^2$ and the codomain of Z is $\mathbf{R}^3$. As an alternative explanation: in order for Bx to make sense, x must be in $\mathbf{R}^2$, and so Bx is in $\mathbf{R}^4$ and $A(Bx)$ is in $\mathbf{R}^3$, therefore the domain of Z is $\mathbf{R}^2$ and the codomain of Z is $\mathbf{R}^3$.

c) True. First note that $(T \circ U)(x) = ABx$, where both A and B have a pivot in every column since T and U are one-to-one. We show that the only solution to $ABx = 0$ is the trivial solution $x = 0$.

Suppose $ABx = 0$. Since A has a pivot in every column, we know that the only vector in the null space of A is the zero vector, so the fact that $A(Bx) = 0$ implies that $Bx = 0$. But B has a pivot in every column, so the fact that $Bx = 0$ gives us $x = 0$. This shows that the only solution to $ABx = 0$ is the zero vector. Therefore, $T \circ U$ is one-to-one.

2. A is $m \times n$ matrix, B is $n \times m$ matrix. Select all correct answers from the box. It is possible to have more than one correct answer.

a) Suppose x is in $\mathbf{R}^m$. Then ABx *must be* in:

Col(A),	Nul(A),	Col(B),	Nul(B)

b) Suppose x in $\mathbf{R}^n$. Then BAx *must be* in:

Col(A),	Nul(A),	Col(B),	Nul(B)

c) If $m > n$, then columns of AB could be linearly $\boxed{independent, \quad dependent}$

d) If $m > n$, then columns of BA could be linearly $\boxed{independent, \quad dependent}$

e) If $m > n$ and $Ax = 0$ has nontrivial solutions, then columns of BA could be linearly $\boxed{independent, \quad dependent}$

Solution.

Recall, AB can be computed as A multiplying every column of B. That is $AB = \begin{pmatrix} Ab_1 & Ab_2 & \cdots Ab_m \end{pmatrix}$ where $B = \begin{pmatrix} b_1 & b_2 & \cdots b_m \end{pmatrix}$.

a) $\boxed{\text{Col}(A)}$. Note Bx is a vector in $\mathbf{R}^n$, so $ABx = A(Bx)$ is multiplying A with a vector in $\mathbf{R}^n$. Therefore, ABx is a linear combination of the columns of A, so ABx must be in Col(A).

b) $\boxed{\text{Col}(B)}$. Similarly, $BAx = B(Ax)$ is multiplying B with a vector in R^m, which is therefore a linear combination of columns of B, so BAx is in Col(B).

c) $\boxed{dependent}$. The fact $m > n$ means A has at most n pivots, so $dim(\text{Col}(A)) \leq n$. From part (a) we know that every vector of the form ABx is in Col(A), which has dimension at most n. This means AB can have at most n pivots. But AB is an $m \times m$ matrix and $m > n$, so AB can't have a pivot in every column and therefore the columns of AB must be linearly dependent.

d) $\boxed{independent, dependent}$. Both are possible. Since $m > n$, we know that A and B have at most n pivots. Here BA is an $n \times n$ matrix, and it is possible (but not guaranteed) for BA to have a pivot in each column. We give two examples below.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{then } BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{then } BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

e) $\boxed{dependent}$. From the second example above, BA has dependent columns, we know "dependent" is one possible answer. Now to see if "independent" is also possible, we need to find out if BA could have n pivots.

Since $Ax = 0$ has nontrivial solution say x^*, then x^* is also a nontrivial solution of $BAx = 0$. That means the equation $BAx = 0$ has at least one free variable, so the columns of BA must be linearly dependent.

To summarize what we are actually study here, there are several relations between these subspaces.
Every vector in Col(AB) is also in Col(A).
Every vector in Col(BA) is also in Col(B).
Every vector in Nul(A) is also in Nul(BA).

Every vector in Nul(B) is also in Nul(AB).

3. Consider the following linear transformations:

$T: \mathbf{R}^3 \longrightarrow \mathbf{R}^2$ T projects onto the xy-plane, forgetting the z-coordinate

$U: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ U rotates clockwise by 90°

$V: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ V scales the x-direction by a factor of 2.

Let A, B, C be the matrices for T, U, V, respectively.

a) Write A, B, and C.

b) Compute the matrix for $V \circ U \circ T$.

c) Describe geometrically the transformation U^{-1} that would undo "U" in the sense that $(U^{-1} \circ U)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. Now, do the same for V. (we will study these in sections 3.5 and 3.6, and they are called "inverses")

Solution.

a) We plug in the unit coordinate vectors:

$$T(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad T(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad T(e_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$U(e_1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad U(e_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$V(e_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad V(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Longrightarrow \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

b) $CBA = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}$.

c) Intuitively, if we wish to "undo" U, we can imagine that we have rotated a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ by 90° clockwise and we want to return the vector back to its original position of $\begin{pmatrix} x \\ y \end{pmatrix}$. To do this, we need to rotate it 90° *counterclockwise*. Therefore, U^{-1} is counterclockwise rotation by 90°.

Similarly, to undo the transformation V that scales the x-direction by 2, we need to scale the x-direction by $1/2$, so V^{-1} scales the x-direction by a factor of $1/2$.

Their matrices are, respectively,

$$B^{-1} = \frac{1}{0 \cdot 0 - (-1) \cdot 1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$C^{-1} = \frac{1}{2 \cdot 1 - 0 \cdot 0} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}.$$