

### Supplemental problems: §3.4

1. Consider  $T: \mathbf{R}^2 \rightarrow \mathbf{R}^3$  defined by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x + y \\ x - y \end{pmatrix}$$

and  $U: \mathbf{R}^3 \rightarrow \mathbf{R}^2$  defined by first projecting onto the  $xy$ -plane (forgetting the  $z$ -coordinate), then rotating counterclockwise by  $90^\circ$ .

- a) Compute the standard matrices  $A$  and  $B$  for  $T$  and  $U$ , respectively.
- b) Compute the standard matrices for  $T \circ U$  and  $U \circ T$ .
- c) Circle all that apply:
 

$T \circ U$ is:	one-to-one	onto
$U \circ T$ is:	one-to-one	onto

#### Solution.

- a) We plug in the unit coordinate vectors to get

$$A = \begin{pmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} | & | & | \\ U(e_1) & U(e_2) & U(e_3) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

- b) The standard matrix for  $T \circ U$  is

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -2 & 0 \\ -1 & -1 & 0 \end{pmatrix}.$$

The standard matrix for  $U \circ T$  is

$$BA = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & 2 \end{pmatrix}.$$

- c) Looking at the matrices, we see that  $T \circ U$  is not one-to-one or onto, and that  $U \circ T$  is one-to-one and onto.

2. Let  $T: \mathbf{R}^3 \rightarrow \mathbf{R}^2$  be the linear transformation which projects onto the  $yz$ -plane and then forgets the  $x$ -coordinate, and let  $U: \mathbf{R}^2 \rightarrow \mathbf{R}^2$  be the linear transformation of rotation counterclockwise by  $60^\circ$ . Their standard matrices are

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

respectively.

- a) Which composition makes sense? (Circle one.)

$$U \circ T \quad T \circ U$$

- b) Find the standard matrix for the transformation that you circled in (b).

**Solution.**

- a) Only  $U \circ T$  makes sense, as the codomain of  $T$  is  $\mathbf{R}^2$ , which is the domain of  $U$ .

- b) The standard matrix for  $U \circ T$  is

$$BA = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 1 & -\sqrt{3} \\ 0 & \sqrt{3} & 1 \end{pmatrix}.$$

3. Find all matrices  $B$  that satisfy

$$\begin{pmatrix} 1 & -3 \\ -3 & 5 \end{pmatrix} B = \begin{pmatrix} -3 & -11 \\ 1 & 17 \end{pmatrix}.$$

**Solution.**

$B$  must have two rows and two columns for the above to compute, so  $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ .

We calculate

$$\begin{pmatrix} 1 & -3 \\ -3 & 5 \end{pmatrix} B = \begin{pmatrix} a - 3c & b - 3d \\ -3a + 5c & -3b + 5d \end{pmatrix}.$$

Setting this equal to  $\begin{pmatrix} -3 & -11 \\ 1 & 17 \end{pmatrix}$  gives us

$$\left. \begin{array}{l} a - 3c = -3 \\ -3a + 5c = 1 \end{array} \right\} \begin{array}{l} \text{solve} \\ \text{~~~~~} \end{array} \quad a = 3, c = 2$$

and

$$\left. \begin{array}{l} b - 3d = -11 \\ -3b + 5d = 17 \end{array} \right\} \begin{array}{l} \text{solve} \\ \text{~~~~~} \end{array} \quad b = 1, d = 4.$$

Therefore,  $B = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ .

4. Let  $T$  and  $U$  be the (linear) transformations below:

$$T(x_1, x_2, x_3) = (x_3 - x_1, x_2 + 4x_3, x_1, 2x_2 + x_3) \quad U(x_1, x_2, x_3, x_4) = (x_1 - 2x_2, x_1).$$

- a) Which compositions makes sense (circle all that apply)?  $U \circ T$   $T \circ U$
- b) Compute the standard matrix for  $T$  and for  $U$ .
- c) Compute the standard matrix for each composition that you circled in (a).

**Solution.**

- a)  $U \circ T$  makes sense, but  $T \circ U$  does not.  
 b) Let  $A$  be the standard matrix for  $T$  and  $B$  be the standard matrix for  $U$ .

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

- c) The matrix for  $U \circ T$  is

$$BA = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -7 \\ -1 & 0 & 1 \end{pmatrix}.$$

5. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.
- a) If  $A$  and  $B$  are matrices and the products  $AB$  and  $BA$  are both defined, then  $A$  and  $B$  must be square matrices with the same number of rows and columns.
- b) If  $A$ ,  $B$ , and  $C$  are nonzero  $2 \times 2$  matrices satisfying  $AB = AC$ , then  $B = C$ .
- c) Suppose  $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$  and  $U : \mathbf{R}^m \rightarrow \mathbf{R}^p$  are one-to-one linear transformations. Then  $U \circ T$  is one-to-one. (What if  $U$  and  $T$  are not necessarily linear?)

### Solution.

- a) False. For example, if  $A$  is any  $2 \times 3$  matrix and  $B$  is any  $3 \times 2$  matrix, then  $AB$  and  $BA$  are both defined.
- b) False. Take  $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ , and  $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Then  $AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$  and  $AC = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ , but  $B \neq C$ .
- c) True. Recall that a transformation  $S$  is one-to-one if  $S(x) = S(y)$  implies  $x = y$  (the same outputs implies the same inputs). Suppose that  $U \circ T(x) = U \circ T(y)$ . Then  $U(T(x)) = U(T(y))$ , so since  $U$  is one-to-one, we have  $T(x) = T(y)$ . Since  $T$  is one-to-one, this implies  $x = y$ . Therefore,  $U \circ T$  is one-to-one. Note that this argument does not use the assumption that  $U$  and  $T$  are linear transformations.

**Alternative:** We'll show that  $U \circ T(x) = 0$  has only the trivial solution. Let  $A$  be the matrix for  $U$  and  $B$  be the matrix for  $T$ , and suppose  $x$  is a vector satisfying  $(U \circ T)(x) = 0$ . In terms of matrix multiplication, this is equivalent to  $ABx = 0$ . Since  $U$  is one-to-one, the only solution to  $Av = 0$  is  $v = 0$ , so if  $A(Bx) = 0$  then  $Bx = 0$ .

Since  $T$  is one-to-one, we know that if  $Bx = 0$  then  $x = 0$ . Therefore, the equation  $(U \circ T)(x) = 0$  has only the trivial solution.

6. In each case, use geometric intuition to either give an example of a matrix with the desired properties or explain why no such matrix exists.
- a) A  $3 \times 3$  matrix  $P$ , which is not the identity matrix or the zero matrix, and satisfies  $P^2 = P$ .
  - b) A  $2 \times 2$  matrix  $A$  satisfying  $A^2 = I$ .
  - c) A  $2 \times 2$  matrix  $A$  satisfying  $A^3 = -I$ .

**Solution.**

- a) Take  $P$  to be the natural projection onto the  $xy$ -plane in  $\mathbf{R}^3$ , so  $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ .

If you apply  $P$  to a vector then the result will be within the  $xy$ -plane of  $\mathbf{R}^3$ , so applying  $P$  a second time won't change anything, hence  $P^2 = P$ .

- b) Take  $A$  to be matrix for reflection across the line  $y = x$ , so  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Since  $A$  swaps the  $x$  and  $y$  coordinates, repeating  $A$  will swap them back to their original positions, so  $AA = I$ .

- c) Note that  $-I$  is the matrix that rotates counterclockwise by  $180^\circ$ , so we need a transformation that will give you counterclockwise rotation by  $180^\circ$  if you do it three times. One such matrix is the rotation matrix for  $60^\circ$  counterclockwise,

$$A = \begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}.$$

Another such matrix is  $A = -I$ .

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**Supplemental problems: §3.5-3.6**

1. a) Fill in:  $A$  and  $B$  are invertible  $n \times n$  matrices, then the inverse of  $AB$  is \_\_\_\_\_.
- b) If the columns of an  $n \times n$  matrix  $Z$  are linearly independent, is  $Z$  necessarily invertible? Justify your answer.
- c) If  $A$  and  $B$  are  $n \times n$  matrices and  $ABx = 0$  has a unique solution, does  $Ax = 0$  necessarily have a unique solution? Justify your answer.

**Solution.**

- a)  $(AB)^{-1} = B^{-1}A^{-1}$ .
- b) Yes. The transformation  $x \rightarrow Zx$  is one-to-one since the columns of  $Z$  are linearly independent. Thus  $Z$  has a pivot in all  $n$  columns, so  $Z$  has  $n$  pivots. Since  $Z$  also has  $n$  rows, this means that  $Z$  has a pivot in every row, so  $x \rightarrow Zx$  is onto. Therefore,  $Z$  is invertible.

Alternatively, since  $Z$  is an  $n \times n$  matrix whose columns are linearly independent, the Invertible Matrix Theorem says that  $Z$  is invertible.

- c) Yes. Since  $AB$  is an  $n \times n$  matrix and  $ABx = 0$  has a unique solution, the Invertible Matrix Theorem says that  $AB$  is invertible. Note  $A$  is invertible and its inverse is  $B(AB)^{-1}$ , since these are square matrices and

$$A(B(AB)^{-1}) = AB(AB)^{-1} = I_n.$$

Since  $A$  is invertible,  $Ax = 0$  has a unique solution by the Invertible Matrix Theorem.

2. Suppose  $A$  is an invertible matrix and

$$A^{-1}e_1 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \quad A^{-1}e_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \quad A^{-1}e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Find  $A$ .

**Solution.**

The columns of  $A^{-1}$  are

$$(A^{-1}e_1 \quad A^{-1}e_2 \quad A^{-1}e_3) \quad \text{so} \quad A = \begin{pmatrix} 4 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To get  $A$  we find  $(A^{-1})^{-1}$ . Row-reducing  $(A^{-1} \mid I)$  eventually gives us

$$\begin{pmatrix} 1 & 0 & 0 & \frac{2}{5} & -\frac{3}{5} & 0 \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad \text{so} \quad A = \begin{pmatrix} \frac{2}{5} & -\frac{3}{5} & 0 \\ -\frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$