Supplemental problems: §5.2

1. True or false. If the statement is always true, answer true and justify why it is true. Otherwise, answer false and give an example that shows it is false.

a) If A and B are $n \times n$ matrices with the same eigenvectors, then A and B have the same characteristic polynomial.

b) If A is a 3×3 matrix with characteristic polynomial $-\lambda^3 + \lambda^2 + \lambda$, then A is invertible.

Solution.

a) False: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ have the same eigenvectors (all nonzero vectors in $\mathbf{R}^2$) but characteristic polynomials λ^2 and $(1-\lambda)^2$, respectively.

b) False: $\lambda = 0$ is a root of the characteristic polynomial, so 0 is an eigenvalue, and A is not invertible.

2. Find all values of a so that $\lambda = 1$ an eigenvalue of the matrix A below.

$$A = \begin{pmatrix} 3 & -1 & 0 & a \\ a & 2 & 0 & 4 \\ 2 & 0 & 1 & -2 \\ 13 & a & -2 & -7 \end{pmatrix}$$

Solution.

We need to know which values of a make the matrix $A - I_4$ noninvertible. We have

$$A - I_4 = \begin{pmatrix} 2 & -1 & 0 & a \\ a & 1 & 0 & 4 \\ 2 & 0 & 0 & -2 \\ 13 & a & -2 & -8 \end{pmatrix}.$$

We expand cofactors along the third column, then the second column:

$$\begin{aligned} \det(A - I_4) &= 2\det\begin{pmatrix} 2 & -1 & a \\ a & 1 & 4 \\ 2 & 0 & -2 \end{pmatrix} \\ &= (2)(1)\det\begin{pmatrix} a & 4 \\ 2 & -2 \end{pmatrix} + (2)(1)\det\begin{pmatrix} 2 & a \\ 2 & -2 \end{pmatrix} \\ &= 2(-2a-8) + 2(-4-2a) = -8a - 24. \end{aligned}$$

This is zero if and only if $a = -3$.

3. If A is an $n \times n$ matrix and $\det(A) = 2$, then 2 is an eigenvalue of A.

Solution.

a) False. For example, $A = \begin{pmatrix} 4 & 0 \\ 0 & 1/2 \end{pmatrix}$ has $\det(A) = 2$ but its eigenvalues are 4 and $\frac{1}{2}$.

4. Let $A = \begin{pmatrix} -3 & 0 & -4 \\ 0 & 3 & 0 \\ 6 & 0 & 7 \end{pmatrix}$.

a) Find the eigenvalues of A.

b) Find a basis for each eigenspace of A. Mark your answers clearly.

c) Is there a basis of $\mathbf{R}^3$ that consists of eigenvectors of A? Justify your answer.

Solution.

a) We solve $0 = \det(A - \lambda I)$.

$$0 = \det\begin{pmatrix} -3-\lambda & 0 & -4 \\ 0 & 3-\lambda & 0 \\ 6 & 0 & 7-\lambda \end{pmatrix} = (3-\lambda)(-1)^4 \det\begin{pmatrix} -3-\lambda & -4 \\ 6 & 7-\lambda \end{pmatrix}$$
$$= (3-\lambda)\left((-3-\lambda)(7-\lambda)+24\right) = (3-\lambda)(\lambda^2 - 4\lambda - 21 + 24)$$
$$= (3-\lambda)(\lambda^2 - 4\lambda + 3) = (3-\lambda)(\lambda-3)(\lambda-1)$$

So $\lambda = 1$ and $\lambda = 3$ are the eigenvalues.

$\underline{\lambda = 1}$: $\left(A - I \mid 0\right) = \left(\begin{array}{ccc|c} -4 & 0 & -4 & 0 \\ 0 & 2 & 0 & 0 \\ 6 & 0 & 6 & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = -R_1/4]{R_3 = R_3 + \frac{3}{2}R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ with solution $x_1 = -x_3$, $x_2 = 0$, $x_3 = x_3$. The 1-eigenspace has basis $\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

$\underline{\lambda = 3}$:

$$\left(A - 3I \mid 0\right) = \left(\begin{array}{ccc|c} -6 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 4 & 0 \end{array}\right) \xrightarrow[\text{then } R_1 = -R_1/6]{R_3 = R_3 + R_1} \left(\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

with solution $x_1 = -\frac{2}{3}x_3$, $x_2 = x_2$, $x_3 = x_3$. The 3-eigenspace has basis $\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2/3 \\ 0 \\ 1 \end{pmatrix}\right\}$.

b) Yes. The eigenvectors that we have found form a basis of $\mathbf{R}^3$. One step of row-reduction shows that the three eigenvectors in $\mathbf{R}^3$ below are linearly independent, and are therefore a basis of $\mathbf{R}^3$ by the Basis Theorem.

$$\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2/3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Supplemental problems: §5.4

1. True or false. Answer true if the statement is always true. Otherwise, answer false.

a) If A is an invertible matrix and A is diagonalizable, then A^{-1} is diagonalizable.

b) A diagonalizable $n \times n$ matrix admits n linearly independent eigenvectors.

c) If A is diagonalizable, then A has n distinct eigenvalues.

Solution.

a) True. If $A = PDP^{-1}$ and A is invertible then its eigenvalues are all nonzero, so the diagonal entries of D are nonzero and thus D is invertible (pivot in every diagonal position). Thus, $A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$.

b) True. By the Diagonalization Theorem, an $n \times n$ matrix is diagonalizable *if and only if* it admits n linearly independent eigenvectors.

c) False. For instance, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is diagonal but has only one eigenvalue.

2. Give examples of 2×2 matrices with the following properties. Justify your answers.

a) A matrix A which is invertible and diagonalizable.

b) A matrix B which is invertible but not diagonalizable.

c) A matrix C which is not invertible but is diagonalizable.

d) A matrix D which is neither invertible nor diagonalizable.

Solution.

a) We can take any diagonal matrix with nonzero diagonal entries:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

b) A shear has only one eigenvalue $\lambda = 1$. The associated eigenspace is the x-axis, so there do not exist two linearly independent eigenvectors. Hence it is not diagonalizable.

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

c) We can take any diagonal matrix with some zero diagonal entries:

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

d) Such a matrix can only have the eigenvalue zero — otherwise it would have two eigenvalues, hence be diagonalizable. Thus the characteristic polynomial is $f(\lambda)=\lambda^2$. Here is a matrix with trace and determinant zero, whose zero-eigenspace (i.e., null space) is not all of $\mathbf{R}^2$:

$$D=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

3. $A=\begin{pmatrix}2 & 3 & 1\\ 3 & 2 & 4\\ 0 & 0 & -1\end{pmatrix}.$

a) Find the eigenvalues of A, and find a basis for each eigenspace.

b) Is A diagonalizable? If your answer is yes, find a diagonal matrix D and an invertible matrix C so that $A=CDC^{-1}$. If your answer is no, justify why A is not diagonalizable.

Solution.

a) We solve $0=\det(A-\lambda I)$.

$$0=\det\begin{pmatrix}2-\lambda & 3 & 1\\ 3 & 2-\lambda & 4\\ 0 & 0 & -1-\lambda\end{pmatrix}=(-1-\lambda)(-1)^6\det\begin{pmatrix}2-\lambda & 3\\ 3 & 2-\lambda\end{pmatrix}=(-1-\lambda)((2-\lambda)^2-9)$$

$$=(-1-\lambda)(\lambda^2-4\lambda-5)=-(\lambda+1)^2(\lambda-5).$$

So $\lambda=-1$ and $\lambda=5$ are the eigenvalues.

$\underline{\lambda=-1}$: $\left(A+I \mid 0\right)=\left(\begin{array}{ccc|c}3 & 3 & 1 & 0\\ 3 & 3 & 4 & 0\\ 0 & 0 & 0 & 0\end{array}\right)\xrightarrow{R_2=R_2-R_1}\left(\begin{array}{ccc|c}3 & 3 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right)\xrightarrow[\text{then } R_1=R_1/3]{R_1=R_1-R_2}$

$\left(\begin{array}{ccc|c}1 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$, with solution $x_1=-x_2$, $x_2=x_2$, $x_3=0$. The (-1)-eigenspace has basis $\left\{\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}\right\}$.

$\underline{\lambda=5}$:

$$\left(A-5I \mid 0\right)=\left(\begin{array}{ccc|c}-3 & 3 & 1 & 0\\ 3 & -3 & 4 & 0\\ 0 & 0 & -6 & 0\end{array}\right)\xrightarrow[R_3=R_3/(-6)]{R_2=R_2+R_1}\left(\begin{array}{ccc|c}-3 & 3 & 1 & 0\\ 0 & 0 & 5 & 0\\ 0 & 0 & 1 & 0\end{array}\right)\xrightarrow[\text{then } R_2\leftrightarrow R_3,\ R_1/(-3)]{R_1=R_1-R_3,\ R_2=R_2-5R_3}\left(\begin{array}{ccc|c}1 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right),$$

with solution $x_1=x_2$, $x_2=x_2$, $x_3=0$. The 5-eigenspace has basis $\left\{\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}\right\}$.

b) A is a 3×3 matrix that only admits 2 linearly independent eigenvectors, so A is not diagonalizable.

4. Let $A = \begin{pmatrix} 8 & 36 & 62 \\ -6 & -34 & -62 \\ 3 & 18 & 33 \end{pmatrix}$.

The characteristic polynomial for A is $\det(A - \lambda I) = -(\lambda-2)^2(\lambda-3)$. Determine whether A is diagonalizable. If it is, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

Solution.

The eigenalues are $\lambda_1 = 3$ and $\lambda_2 = 2$.

For $\lambda_1 = 3$, we row-reduce $A - 3I$:

$$\begin{pmatrix} 5 & 36 & 62 \\ -6 & -37 & -62 \\ 3 & 18 & 30 \end{pmatrix} \xrightarrow[(\text{New } R_1)/3]{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 6 & 10 \\ -6 & -37 & -62 \\ 5 & 36 & 62 \end{pmatrix} \xrightarrow[R_3 = R_3 - 5R_1]{R_2 = R_2 + 6R_1} \begin{pmatrix} 1 & 6 & 10 \\ 0 & -1 & -2 \\ 0 & 6 & 12 \end{pmatrix}$$

$$\xrightarrow[\text{then } R_2 = -R_2]{R_3 = R_3 + 6R_2} \begin{pmatrix} 1 & 6 & 10 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 = R_1 - 6R_2} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the solutions to $\left(A - 3I \mid 0\right)$ are $x_1 = 2x_3,\ x_2 = -2x_3,\ x_3 = x_3$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}. \qquad \text{The 3-eigenspace has basis } \left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

For $\lambda_2 = 2$, we row-reduce $A - 2I$:

$$\begin{pmatrix} 6 & 36 & 62 \\ -6 & -36 & -62 \\ 3 & 18 & 31 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 6 & \frac{31}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The solutions to $\left(A - 2I \quad 0\right)$ are $x_1 = -6x_2 - \frac{31}{3}x_3,\ x_2 = x_2,\ x_3 = x_3$.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -6x_2 - \frac{31}{3}x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix}.$$

The 2-eigenspace has basis $\left\{ \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{31}{3} \\ 0 \\ 1 \end{pmatrix} \right\}$.

Therefore, $A = CDC^{-1}$ where

$$C = \begin{pmatrix} 2 & -6 & -\frac{31}{3} \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Note that we arranged the eigenvectors in C in order of the eigenvalues $3,\ 2,\ 2$, so we had to put the diagonals of D in the same order.

5. Which of the following 3×3 matrices are necessarily diagonalizable over the real numbers? (Circle all that apply.)

1. A matrix with three distinct real eigenvalues.
2. A matrix with one real eigenvalue.
3. A matrix with a real eigenvalue λ of algebraic multiplicity 2, such that the λ-eigenspace has dimension 2.
4. A matrix with a real eigenvalue λ such that the λ-eigenspace has dimension 2.

Solution.

The matrices in 1 and 3 are diagonalizable. A matrix with three distinct real eigenvalues automatically admits three linearly independent eigenvectors. If a matrix A has a real eigenvalue λ_1 of algebraic multiplicity 2, then it has another real eigenvalue λ_2 of algebraic multiplicity 1. The two eigenspaces provide three linearly independent eigenvectors.

The matrices in 2 and 4 need not be diagonalizable.

6. Suppose a 2×2 matrix A has eigenvalue $\lambda_1 = -2$ with eigenvector $v_1 = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$, and eigenvalue $\lambda_2 = -1$ with eigenvector $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

a) Find A.

b) Find A^{100}.

Solution.

a) We have $A = CDC^{-1}$ where

$$C = \begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

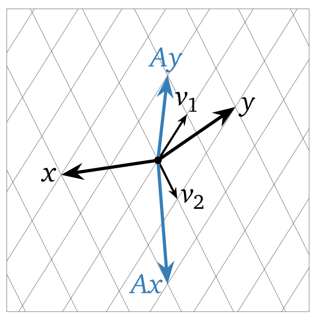
We compute $C^{-1} = \dfrac{1}{-5/2}\begin{pmatrix} -1 & -1 \\ -1 & 3/2 \end{pmatrix} = \dfrac{1}{5}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix}$.

$$A = CDC^{-1} = \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -8 & -3 \\ -2 & -7 \end{pmatrix}.$$

b)

$$\begin{aligned} A^{100} = CD^{100}C^{-1} &= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix} \cdot D^{100} \begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2^{100} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3/2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2\cdot 2^{100} & 2\cdot 2^{100} \\ 2 & -3 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 3\cdot 2^{100}+2 & 3\cdot 2^{100}-3 \\ 2^{101}-2 & 2^{101}+3 \end{pmatrix}. \end{aligned}$$

7. Suppose that $A = C\begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}C^{-1}$, where C has columns v_1 and v_2. Given x and y in the picture below, draw the vectors Ax and Ay.



Solution.

A does the same thing as $D = \begin{pmatrix} 1/2 & 0 \\ 0 & -1 \end{pmatrix}$, but in the v_1, v_2-coordinate system. Since D scales the first coordinate by $1/2$ and the second coordinate by -1, hence A scales the v_1-coordinate by $1/2$ and the v_2-coordinate by -1.