MATH 1553, SPRING 2024
SAMPLE MIDTERM 1A: SOLUTIONS

Name		**GT ID**	

Circle your instructor and lecture below. Some professors teach more than one lecture, so be sure to circle the correct choice!

Jankowski (A and HP, 8:25-9:15 AM) Jankowski (G, 12:30-1:20 PM)

Hausmann (I, 2:00-2:50 PM) Sanchez-Vargas (M, 3:30-4:20 PM)

Athanasouli (N and PNA, 5:00-5:50 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. We recommend completing the practice exam in 75 minutes, without notes or distractions.

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1. TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

a) Suppose a system of linear equations corresponds to an augmented matrix whose RREF has **bottom** row equal to

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \end{array}\right).$$

Then the system must be consistent.

$\boxed{\text{TRUE}}$ FALSE

b) If v_1 and v_2 are vectors in $\mathbf{R}^n$, then the vector $3v_1 - v_2$ is in $\text{Span}\{v_1, v_2\}$.

$\boxed{\text{TRUE}}$ FALSE

c) If v_1, v_2, and v_3 are vectors in $\mathbf{R}^2$, then the vector equation

$$x_1v_1 + x_2v_2 + x_3v_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

must have infinitely many solutions.

$\boxed{\text{TRUE}}$ FALSE

d) Suppose A is a 3×2 matrix and b is a vector in $\mathbf{R}^3$ so that $Ax = b$ has exactly one solution. Then the only solution to the homogeneous equation $Ax = 0$ is the trivial solution.

$\boxed{\text{TRUE}}$ FALSE

e) If A is a 4×5 matrix and the solution set to $Ax = 0$ is a line, then $Ax = b$ must be inconsistent for some b in $\mathbf{R}^4$.

TRUE $\boxed{\text{FALSE}}$

Solution.

a) True. Since that is the bottom row of the RREF, this means there is no pivot in the rightmost column.

b) True. This follows directly from the definition of span.

c) True. The matrix $\left(\begin{array}{ccc|c} v_1 & v_2 & v_3 & 0 \end{array}\right)$ must have at least one column without a pivot to the left of the augment bar, so the system has infinitely many solutions.

d) True. The solution set to $Ax = 0$ is a translation of the solution set to the consistent equation $Ax = b$ (and vice versa), which is a single point. Therefore, $Ax = 0$ has exactly one solution, namely the trivial solution.

e) False. For example, the 4×5 matrix A below has the property that the solution set to $Ax = 0$ is a line but $Ax = b$ is consistent for each b in $\mathbf{R}^4$.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

2. Multiple choice and short answer. You do not need to show work or justify your answers. Parts (a), (b), (c), and (d) are unrelated.

a) (3 points) Which of the following equations are linear equations in the variables x, y, and z? Clearly circle LINEAR or NOT LINEAR in each case.

(i) $\ln(4)x - y + z = \sqrt{11}$. $\boxed{\text{LINEAR}}$ NOT LINEAR

(ii) $x - y^2 - z = 0$. LINEAR $\boxed{\text{NOT LINEAR}}$

(iii) $3x - y + 2z = 1$. $\boxed{\text{LINEAR}}$ NOT LINEAR

b) (2 points) Which of the following matrices are in reduced row echelon form (RREF)? Clearly circle all that apply.

(i) $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 0 & 0 & 1 \end{array}\right)$

$\boxed{\text{(ii)}}$ $\left(\begin{array}{cccc|c} 1 & 5 & -3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

c) (2 pts) Find all values of h (if there are any) so that the following matrix is in RREF.

$$\left(\begin{array}{ccc|c} 1 & -3 & 0 & 1 \\ 0 & 1 & h & -5 \end{array}\right).$$

Clearly circle the one correct answer below.

(i) $h = 0$ only

(ii) $h = 1$ only

(iii) All real values of h except 0

(iv) All real values of h except 1

(v) All real values of h

$\boxed{\text{(vi)}}$ The matrix is not in RREF, no matter what h is.

d) (3 points) Suppose v_1, v_2, and b are vectors in $\mathbf{R}^3$. Which of the following are true? Clearly circle all that apply.

(i) The vector equation $x_1v_1 + x_2v_2 = b$ corresponds to a system of two linear equations in three variables.

$\boxed{\text{(ii)}}$ If the vector equation $x_1v_1 + x_2v_2 = b$ has a solution, then some vector w in $\mathbf{R}^3$ is not in Span$\{v_1, v_2, b\}$.

$\boxed{\text{(iii)}}$ If $x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a solution to the equation $x_1v_1 + x_2v_2 = b$, then $b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

3. Short answer and multiple choice. Parts (a), (b), (c), and (d) are unrelated. Briefly show work on part (b).

a) (2 points) The linear system

$$\begin{aligned} 5x - y + 3z - 2w &= 4 \\ -x + 3y + z - 4w &= 2 \\ x + 0y + z - 4w &= -1 \end{aligned}$$

is consistent. How many solutions does this system have? Circle the one answer that gives the exact number of solutions.

(i) 0 solutions

(ii) 1 solution

(iii) 2 solutions

$\boxed{\text{(iv)}}$ Infinitely many solutions

b) (2 points) Compute the product $\begin{pmatrix} -5 & 2 \\ 0 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

$$\begin{pmatrix} -5 & 2 \\ 0 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = -1\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 9 \\ 6 \\ 1 \end{pmatrix}}.$$

c) (2 points) A consistent system of four linear equations in three variables corresponds to an augmented matrix whose RREF has two pivots. Complete the following statements by clearly circling the one correct answer in each case.

(i) The solution set for the system is:

a point $\quad$ $\boxed{\text{a line}}$ $\quad$ a plane $\quad$ all of $\mathbf{R}^3$ $\quad$ all of $\mathbf{R}^4$

(ii) Each solution to the system of linear equations is in:

$\mathbf{R}$ $\quad$ $\mathbf{R}^2$ $\quad$ $\boxed{\mathbf{R}^3}$ $\quad$ $\mathbf{R}^4$

d) (4 points) Suppose A is a 3×4 matrix. Which of the following are true? Circle all that apply.

$\boxed{\text{(i)}}$ The homogeneous equation $Ax = 0$ must have infinitely many solutions.

(ii) If v is in the column span of A, then v is in $\mathbf{R}^4$.

(iii) If A has a pivot in its rightmost column, then the equation $Ax = 0$ is inconsistent.

(iv) The equation $Ax = b$ must be inconsistent for some vector b in $\mathbf{R}^3$.

4. **a)** (2 pts) Find the value of k so that $\begin{pmatrix}10\\k\end{pmatrix}$ is a linear combination of $\begin{pmatrix}2\\-1\end{pmatrix}$ and $\begin{pmatrix}-4\\2\end{pmatrix}$.

Write your answer here: $\boxed{k=-5}$.

This is a quick row-reduction of $\left(\begin{array}{cc|c}2 & -4 & 10\\ -1 & 2 & k\end{array}\right) \xrightarrow[\text{then } R_2=R_2+R_1]{R_1=R_1/2} \left(\begin{array}{cc|c}1 & -2 & 5\\ 0 & 0 & k+5\end{array}\right)$ to find $k=-5$ for the system to be consistent. Alternatively, we can see that $\begin{pmatrix}10\\k\end{pmatrix}$ is on the line spanned by $\begin{pmatrix}2\\-1\end{pmatrix}$ and $\begin{pmatrix}-4\\2\end{pmatrix}$ when it is a scalar multiple of the vectors, which is when $k=-5$.

b) (3 points) Write three different vectors v_1, v_2, and v_3 in $\mathbf{R}^3$ that satisfy both of the following conditions.

(i) The span of any two of the vectors is a plane.

(ii) Span$\{v_1, v_2, v_3\}$ is also a plane.

Write your answer here: Many possibilities. For example,

$$v_1=\begin{pmatrix}1\\0\\0\end{pmatrix}, \quad v_2=\begin{pmatrix}0\\1\\0\end{pmatrix}, v_3=\begin{pmatrix}1\\1\\0\end{pmatrix}.$$

c) (2 points) Let $A=\begin{pmatrix}4 & 12 & 16\\ -1 & -3 & -4\end{pmatrix}$.

Write one **nonzero** vector b so that the equation $Ax=b$ is consistent.

Write your answer here: All three columns of A are scalar multiples of $\begin{pmatrix}4\\-1\end{pmatrix}$, so we must write a nonzero scalar multiple of $\begin{pmatrix}4\\-1\end{pmatrix}$ as our answer. For example,

$\begin{pmatrix}4\\-1\end{pmatrix}, \begin{pmatrix}-4\\1\end{pmatrix}, \begin{pmatrix}12\\-3\end{pmatrix}, \begin{pmatrix}16\\-4\end{pmatrix}, \begin{pmatrix}20\\-5\end{pmatrix}$, etc.

d) (3 points) Let $v_1=\begin{pmatrix}1\\-2\\1\end{pmatrix}$ and $v_2=\begin{pmatrix}2\\-4\\2\end{pmatrix}$. Which of the following statements are true? Clearly select all that apply.

(i) Span$\{v_1, v_2\}$ is a plane in $\mathbf{R}^3$.

(ii) The vector $w=\begin{pmatrix}1\\-2\\0\end{pmatrix}$ is a linear combination of v_1 and v_2.

$\boxed{\text{(iii)}}$ If b is a vector and the vector equation

$$x_1v_1+x_2v_2=b$$

is consistent, then the solution set is a line in $\mathbf{R}^2$.

The rest of the exam is free response. Show your work from here onward! A correct answer without sufficient work will receive little or no credit.

5. (10 points) Consider the system of linear equations in x and y given by

$$x - hy = 5$$
$$6x + 12y = k,$$

where h and k are real numbers.

Before doing any of the three parts, we do one step of row-reduction.

$$\left(\begin{array}{cc|c} 1 & -h & 5 \\ 6 & 12 & k \end{array}\right) \xrightarrow{R_2 = R_2 - 6R_1} \left(\begin{array}{cc|c} 1 & -h & 5 \\ 0 & 12+6h & k-30 \end{array}\right).$$

a) Find all values of h and k (if there are any) so that the system is inconsistent.

The system is inconsistent when the rightmost column has a pivot, thus $12+6h = 0$ and $k - 30 \neq 0$.

$$\boxed{h = -2, \qquad k \neq 30}.$$

b) Find all values of h and k (if there are any) so that the system has exactly one solution.

The system has a unique solution when the system has two pivots and they are both to the left of the augment bar, so $12 + 6h \neq 0$ and k can be anything.

$$\boxed{h \neq -2, \qquad k \text{ any real number}}.$$

c) Find all values of h and k (if there are any) so that the system has infinitely many solutions.

The system has infinitely many solutions when the rightmost column does not have a pivot and some other column (in this case, the second) does not either. This means the second row is all zeros, so $12 + 6h = 0$ and $k - 30 = 0$.

$$\boxed{h = -2, \qquad k = 30}.$$

Free response. Show your work!

6. Consider the following linear system of equations in the variables $x_1,\ x_2,\ x_3,\ x_4$:

$$x_1 - 2x_2 - x_3 + x_4 = 1$$
$$-2x_1 + 4x_2 + 3x_3 - 2x_4 = -5$$
$$4x_1 - 8x_2 - 4x_3 + 4x_4 = 4.$$

a) (4 points) Write the augmented matrix corresponding to this system, and put the augmented matrix into RREF.

$$\left(\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 1 \\ -2 & 4 & 3 & -2 & -5 \\ 4 & -8 & -4 & 4 & 4 \end{array}\right) \xrightarrow[R_3=R_3-4R_1]{R_2=R_2+2R_1} \left(\begin{array}{cccc|c} 1 & -2 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1=R_1+R_2} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

b) (4 points) The system is consistent. Write the set of solutions to the system of equations in parametric vector form.

From the RREF, we see x_2 and x_4 are free, and $x_1 - 2x_2 + x_4 = -2$, and $x_3 = -3$. For x_1 we isolate to get $x_1 = -2 + 2x_2 - x_4$, so

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 + 2x_2 - x_4 \\ x_2 \\ -3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_4 \\ 0 \\ 0 \\ x_4 \end{pmatrix}$$
$$= \begin{pmatrix} -2 \\ 0 \\ -3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

c) (2 points) Write *one* nonzero vector x that solves the corresponding **homogeneous** system of equations below. Briefly justify your answer.

$$x_1 - 2x_2 - x_3 + x_4 = 0$$
$$-2x_1 + 4x_2 + 3x_3 - 2x_4 = 0$$
$$4x_1 - 8x_2 - 4x_3 + 4x_4 = 0.$$

By the theory in section 2.4, the homogeneous solutions are all vectors in Span $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Therefore, possible answers are $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, etc.

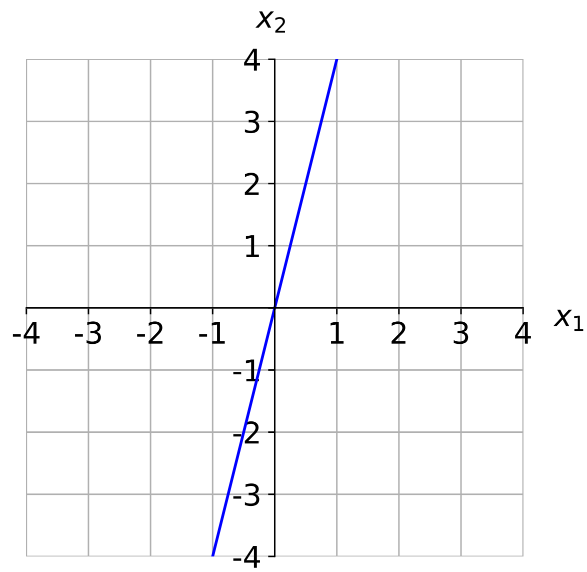
The zero vector is not an answer, since we are explicitly asked for a nonzero vector.

7. Show your work! Parts (a) and (b) are unrelated.

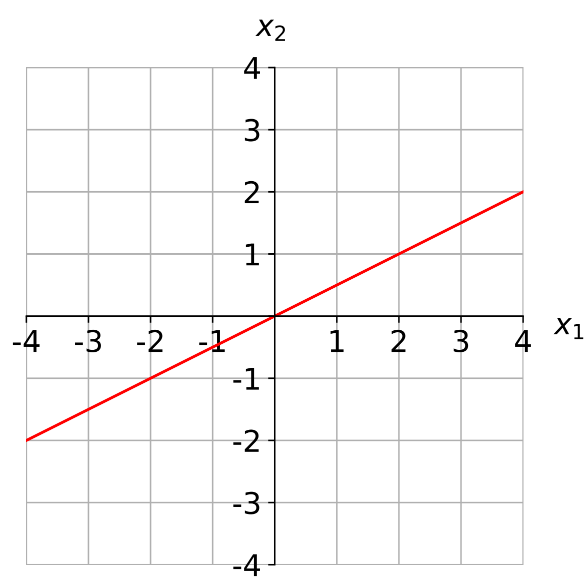
a) Let $A = \begin{pmatrix} 1 & -2 \\ 4 & -8 \end{pmatrix}$.

(i) (2 points) Draw the span of the columns of A on the graph below.

The span of $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -8 \end{pmatrix}$ is just the span of $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, which is the line through the origin containing the point $(1, 4)$.



(ii) (4 points) Draw the solution set for $Ax = 0$ on the graph below.



$\left(A \,\middle|\, 0\right)$ row-reduces to $\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ so $x_1 - 2x_2 = 0$, thus $x_1 = 2x_2$ and x_2 is free, so our solution set is the line $x_2 = \dfrac{x_1}{2}$. Alternatively, if we wanted to write this in parametric vector form, we would get $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, so the solution set is the span of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

b) (4 points) Write an augmented matrix in RREF that corresponds to a system of linear equations in the variables x_1, x_2, and x_3 whose solution set has parametric form

$$x_1 = -1 + 2x_2, \qquad x_2 = x_2 \ \ (x_2 \text{ real}), \qquad x_3 = 0.$$

We need three columns to the left of the augment bar because we have three variables, and we need $x_1 - 2x_2 = -1$ with x_2 free and $x_3 = 0$. We can express this by $\left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$ or by equivalent augmented matrices, for example

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad \left(\begin{array}{ccc|c} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \text{etc.}$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.