MATH 1553, FINAL EXAMINATION SOLUTIONS
SPRING 2024

Name		**GT ID**	

Circle your instructor and lecture below. Some professors teach more than one lecture, so be sure to circle the correct choice!

Jankowski (A and HP, 8:25-9:15 AM) Jankowski (G, 12:30-1:20 PM)

Hausmann (I, 2:00-2:50 PM) Sanchez-Vargas (M, 3:30-4:20 PM)

Athanasouli (N and PNA, 5:00-5:50 PM) OR: *Advanced Standing Student*

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 100 points, and you have 170 minutes to complete this exam. Each problem is worth 10 points.
- Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 8:50 PM on Tuesday, April 30.

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Problem 1. [1 point each]

Solutions are on the next page. As stated in the instructions, **the entries of all matrices on the exam are real numbers** unless stated otherwise.

a) **T** **F** Suppose $\{v_1, v_2, v_3, v_4, v_5\}$ is a basis for $\mathbf{R}^n$. Then $n = 5$.

b) **T** **F** Suppose $T : \mathbf{R}^{20} \to \mathbf{R}^7$ is a linear transformation with standard matrix A, so $T(x) = Ax$. Then $\dim(\text{Nul } A) \geq 13$.

c) **T** **F** Let A be an $m \times n$ matrix, and let T be the corresponding matrix transformation $T(x) = Ax$. If $m > n$, then T cannot be onto.

d) **T** **F** The set $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \ : \ x - y + z = 3 \right\}$ is a subspace of $\mathbf{R}^3$.

e) **T** **F** There is a 3×3 matrix A, whose entries are real numbers, so that $2 - i$ and $3i$ are eigenvalues of A.

f) **T** **F** Every nonzero vector in $\mathbf{R}^3$ is an eigenvector of the 3×3 identity matrix.

g) **T** **F** Suppose that u and v are vectors in the 4-eigenspace of some $n \times n$ matrix A. Then $4u - 3v$ must also be in the 4-eigenspace of A.

h) **T** **F** Let A be a 3×3 matrix with characteristic polynomial

$$\det(A - \lambda I) = (1 - \lambda)(3 - \lambda)^2,$$

and suppose $A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ and $A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$.

Then A must be diagonalizable.

i) **T** **F** Suppose A is an $m \times n$ matrix and b is a vector in the column space of A. Then every solution to $Ax = b$ is also a least squares solution to $Ax = b$.

j) **T** **F** Suppose that W is a subspace of $\mathbf{R}^n$ and that u is a vector in W. Then the orthogonal projection of u onto W is the zero vector.

Solutions to Problem 1.

a) True: $\mathbf{R}^n$ has a basis with 5 vectors, so $\dim(\mathbf{R}^n) = 5$, therefore $n = 5$.

b) True: A is 7×20, Col(A) lives in $\mathbf{R}^7$, and

$$\dim(\text{Col } A) + \dim(\text{Nul } A) = 20 \text{ (by the Rank Theorem)},$$

so $\dim(\text{Nul } A) \geq 13$.

c) True: A would have more rows than columns, so A could not have a pivot in every row.

d) False: W does not even contain the zero vector, since $0 - 0 + 0 \neq 3$.

e) False: if $2 - i$ and $3i$ are eigenvalues, then so are their complex conjugates $2 + i$ and $-3i$, therefore A would have 4 different eigenvalues which is impossible for a 3×3 matrix.

f) True: $Ix = x$ for every x in $\mathbf{R}^n$, so every nonzero vector in $\mathbf{R}^n$ is an eigenvector of I corresponding to $\lambda = 1$.

g) True: eigenspaces are subspaces, so if u and v are in the 4-eigenspace of A then so is $4u - 3v$.

h) True: A is a 3×3 matrix with eigenvalues 1 and 3. We are given that $\lambda = 3$ has geometric multiplicity 2, and Since $\lambda = 1$ automatically has geometric multiplicity 1 (since it has algebraic multiplicity 1), we conclude that the real geometric multiplicities sum to 3, therefore A is diagonalizable.

i) True: the fact that b is in Col(A) means that $b = b_{\text{Col}(A)}$, so every solution to $Ax = b$ is also a solution to $Ax = b_{\text{Col}(A)}$ and vice versa. This problem was copied from the Studypalooza problems list and the 6.5 Webwork.

j) False: if u is in W, then the orthogonal projection of u onto W is u.

Problem 2.

Solutions are on the next page.

a) (3 points) Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; x^2 + y^2 \leq 5 \right\}$. Answer the following questions.

(i) Does V contain the zero vector? $\boxed{\text{YES}}$ NO

(ii) Is V closed under addition? In other words, if u and v are in V, must it be true that $u + v$ is in V? YES $\boxed{\text{NO}}$

(iii) Is V closed under scalar multiplication? In other words, if c is a real number and u is in V, must it be true that cu is in V? YES $\boxed{\text{NO}}$

b) (3 points) Suppose that A is a 2024×100 matrix.
Which of the following are **possible**? Clearly circle all that apply.

(i) The dimension of Row(A) is 105.

(ii) The dimension of Nul(A) is 105.

(iii) $\boxed{\text{The transformation } T(x) = Ax \text{ is one-to-one.}}$

c) (2 points) Let A be a 35×50 matrix that has 20 pivots. Which one of the following describes the null space of A? Clearly circle your answer.

(i) Nul(A) is a 30-dimensional subspace of $\mathbf{R}^{35}$.

(ii) Nul(A) is a 20-dimensional subspace of $\mathbf{R}^{50}$.

(iii) Nul(A) is a 15-dimensional subspace of $\mathbf{R}^{35}$.

(iv) $\boxed{\text{Nul}(A) \text{ is a 30-dimensional subspace of } \mathbf{R}^{50}.}$

d) (2 pts) Let $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\}$, and let $v = \begin{pmatrix} 2 \\ 1 \\ 1 \\ -1 \end{pmatrix}$.

Which of the following statements are true? Clearly circle all that apply.

(i) The set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ is a basis for W.

(ii) $\boxed{v \text{ is in } W^{\perp}.}$

Solutions to Problem 2.

a) Just drawing the picture, we see V is the circle of radius $\sqrt{5}$ centered at the origin, so it contains the zero vector but is not closed under addition or scalar multiplication.
(i) Yes: $0^2 + 0^2 \leq 5$.
(ii) No: for example, if $u = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ then u and v are in V. However, $u + v = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ which is not in V since $2^2 + 2^2 = 8 > 5$.
(iii) No: for example, if $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then u is in V, but $10v$ is not in V since $10^2+0^2 > 5$.

b) By the Rank Theorem, dim(Col A)+dim(Nul A) $= 100$, so the largest that dim(Col A) or dim(Col A) can be is 100.
(i) Not possible, since dim(Row A) = dim(Col A) and dim(Col A) ≤ 100.
(ii) Not possible, since dim(Nul A) ≤ 100.
(iii) Possible: A is 2024×100 so it is possible for A to have 100 pivots, in which case A has a pivot in every column and its corresponding transformation $T(x) = Ax$ is one-to-one.

c) Since the 35×50 matrix A has 20 pivots, we know that dim(Col A) = 20 and that Nul$A$ is a subspace of $\mathbf{R}^{50}$. By the Rank Theorem,

$$\dim(\text{Col } A) + \dim(\text{Nul } A) = 50, \qquad 20 + \dim(\text{Nul } A) = 50, \qquad \dim(\text{Nul } A) = 30.$$

d) (i) False: for example, Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ does not contain the vector $\begin{pmatrix} 1 \\ -1 \\ 3 \\ 4 \end{pmatrix}$ which is in W.
(ii) True: $v \cdot \begin{pmatrix} 1 \\ -1 \\ 3 \\ 4 \end{pmatrix} = 2 - 1 + 3 - 4 = 0$ and $v \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} = 1 + 2 - 3 = 0$. By properties of the dot product, this means that v is orthogonal to Span $\left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\}$.

Problem 3.

Solutions are on the next page.

a) (4 points) Suppose $T : \mathbf{R}^a \to \mathbf{R}^b$ is a linear transformation. Which of the following conditions guarantee that T is onto? Clearly circle **all** that apply.

(i) For each y in $\mathbf{R}^b$, there is at least one x in $\mathbf{R}^a$ so that $T(x) = y$.

(ii) For each x in $\mathbf{R}^a$, there is at most one y in $\mathbf{R}^b$ so that $T(x) = y$.

(iii) For each x in $\mathbf{R}^a$, there is exactly one y in $\mathbf{R}^b$ so that $T(x) = y$.

(iv) For each y in $\mathbf{R}^b$, there is exactly one x in $\mathbf{R}^a$ so that $T(x) = y$.

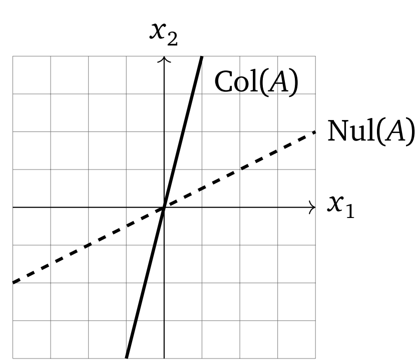
b) (3 points) Which of the following matrices A are invertible? Clearly circle **all** that apply.

(i) $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$

(ii) Any 3×3 matrix A that has eigenvalues $\lambda = 1$, $\lambda = -1$, and $\lambda = 3$.

(iii) The 2×2 matrix A that rotates vectors in $\mathbf{R}^2$ by 60 degrees counterclockwise.

c) (3 points) Write a matrix A so that Col(A) is the **solid** line below and Nul(A) is the **dashed** line below.



Many answers possible, for example $A = \begin{pmatrix} 1 & -2 \\ 4 & -8 \end{pmatrix}$

Solutions to Problem 3.

a) Statement (ii) is almost the definition of "transformation" but not quite, whereas part (iii) is nearly word-for-word the definition of transformation. Neither guarantees that T is onto, since the linear transformation $T(x, y) = (x, y, 0)$ satisfies statements (ii) and (iii) but is not onto.

Statement (i) is nearly word-for-word the definition of onto, while (iv) means that T is invertible which means that T is both one-to-one and onto.

b) In (i), the matrix A is not invertible because one step of row-reduction gives a row of zeros. Alternatively, we could compute that $\det(A) = 0$.

For (ii): if A is a 3×3 matrix eigenvalues 1, -1, and 3, then these are the only eigenvalues of A since an $n \times n$ matrix cannot have more than n different eigenvalues. Consequently we know 0 is not an eigenvalue, so A must be invertible.

(iii) is invertible. Either we could write the matrix A and calculate that its determinant is not zero, or we could observe that A is invertible by the Invertible Matrix Theorem because the only vector that satisfies $Ax = 0$ is the zero vector.

c) We need $\text{Col}(A) = \text{Span}\left\{\begin{pmatrix}1\\4\end{pmatrix}\right\}$, so both columns of A must be multiples of $\begin{pmatrix}1\\4\end{pmatrix}$. We also need

$$\text{Nul}(A) = \text{Span}\left\{\begin{pmatrix}2\\1\end{pmatrix}\right\},$$

which means that the parametric vector form for the solution set of $Ax = 0$ is $x_2\begin{pmatrix}2\\1\end{pmatrix}$, or in other words $x_1 = 2x_2$, so $x_1 - 2x_2 = 0$.
The RREF of $(A \mid 0)$ is therefore $\left(\begin{array}{cc|c}1 & -2 & 0\\0 & 0 & 0\end{array}\right)$.

Many answers are possible for A, for example:

$$A = \begin{pmatrix}1 & -2\\4 & -8\end{pmatrix} \qquad A = \begin{pmatrix}2 & -4\\8 & -16\end{pmatrix}, \qquad A = \begin{pmatrix}-1 & 2\\-4 & 8\end{pmatrix}, \qquad \text{etc.}$$

Problem 4.

Solutions are on the next page.

a) (2 points) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1$. Find $\det\begin{pmatrix} d & e & f \\ 4d-3a & 4e-3b & 4f-3c \\ g & h & i \end{pmatrix}$.

Clearly circle your answer below.

(i) 1 (ii) −1 **[(iii) 3]** (iv) −3 (v) 4 (vi) −4

(vii) 12 (viii) −12 (ix) −24 (x) none of these

b) (2 points) Find the area of the triangle with vertices $(-4,4)$, $(2,-2)$, and $(6,-1)$.

(i) 3/2 (ii) 3 (iii) 15/2 **[(iv) 15]** (v) 30 (vi) 45/2

(vii) 45 (viii) 50 (ix) 90 (x) none of these

c) (2 points) Suppose A and B are 3×3 matrices satisfying $\det(A) = 4$ and $\det(B) = 2$. Find $\det(-3A^{-1}B)$.

(i) −3/2 (ii) 9/2 **[(iii) −27/2]** (iv) −24 (v) 27/2

(vi) −72 (vii) −9/2 (viii) 72 (ix) none of these

d) (4 points) Suppose A is a 4×4 matrix with characteristic polynomial

$$\det(A-\lambda I) = (2-\lambda)^2(3-\lambda)(-1-\lambda).$$

Which of the following statements are true? Clearly circle all that apply.

[(i)] A is invertible.

[(ii)] If the 2-eigenspace of A is a plane, then A must be diagonalizable.

(iii) It is possible that the 3-eigenspace of A is a plane.

[(iv)] The zero vector is **not** an eigenvector of A.

Solutions to Problem 4.

a) The original matrix has determinant 1. To get from there to the final matrix, we do one row swap, one row scale by a factor of −3, and one row replacement. The final answer is therefore $(1)(-1)(-3)=3$.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix} \quad \text{(new det is -1)}$$

$$\xrightarrow{R_2 = -3R_2} \begin{pmatrix} d & e & f \\ -3a & -3b & -3c \\ g & h & i \end{pmatrix} \quad \text{(new det is 3)}$$

$$\xrightarrow{R_2 = R_2 + 4R_1} \begin{pmatrix} d & e & f \\ 4d-3a & 4e-3b & 4f-3c \\ g & h & i \end{pmatrix}. \quad \text{(det is still 3)}$$

b) The vector from $(-4,4)$ to $(2,-2)$ is $v_1 = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$. The vector from $(-4,4)$ to $(6,-1)$ is $v_2 = \begin{pmatrix} 10 \\ -5 \end{pmatrix}$. We take half the area of the parallelogram determinated by v_1 and v_2:

$$\frac{1}{2}\left|\det\begin{pmatrix} 6 & 10 \\ -6 & -5 \end{pmatrix}\right| = \frac{1}{2}|-30+60| = 15.$$

c) We use properties of determinants:

$$\begin{aligned} \det(-3A^{-1}B) &= (-3)^3\det(A^{-1}B) = -27\det(A^{-1})\det(B) \\ &= -27\frac{1}{\det(A)}\det(B) = -27\cdot\frac{1}{4}\cdot 2 = -\frac{27}{2}. \end{aligned}$$

d) (i) True, the eigenvalues of A are 2, 3, and −1, so 0 is not an eigenvalue of A.

(ii) True, because if the 2-eigenspace is a plane, then from the other two different eigenvalues we get a sum of geometric multiplicities of 4, therefore A is diagonalizable.

(iii) False: the 3-eigenspace of A cannot be a plane, because $\lambda = 3$ has algebraic multiplicity 1 and therefore geometric multiplicity 1 (geometric multiplicity can never be larger than algebraic multiplicity).

(iv) True: the zero vector can never be an eigenvector of any square matrix A.

Problem 5.

Solutions are on the next page.

a) (3 points) Let A be a 2×2 matrix whose entries are real numbers. Which of the following statements must be true? Clearly circle all that apply.

$\boxed{\text{(i)}}$ If A has $\lambda = -5$ as an eigenvalue with algebraic multiplicity 2, then -5 is the only eigenvalue of A.

$\boxed{\text{(ii)}}$ If $\lambda = 1-4i$ is an eigenvalue of A, then A does not have any real eigenvalues.

(iii) If A is a stochastic matrix, then the only eigenvalue of A is $\lambda = 1$.

b) (3 points) Let A be the 2×2 matrix that reflects vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ across the line $y = 3x$. Which of the following are true? Clearly circle all that apply.

(i) The eigenvalues of A are $\lambda = 0$ and $\lambda = 1$.

$\boxed{\text{(ii)}}$ A is diagonalizable.

$\boxed{\text{(iii)}}$ $A\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

c) (2 points) Let $A = \begin{pmatrix} 0.3 & 0.1 \\ 0.7 & 0.9 \end{pmatrix}$. It has the property that $A\begin{pmatrix} 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$.

What vector does $A^n\begin{pmatrix} 400 \\ 0 \end{pmatrix}$ approach as n gets large? Clearly circle your answer.

(i) $\begin{pmatrix} 64 \\ 336 \end{pmatrix}$ (ii) $\begin{pmatrix} 1/8 \\ 7/8 \end{pmatrix}$ $\boxed{\text{(iii)}}$ $\begin{pmatrix} 50 \\ 350 \end{pmatrix}$ (iv) $\begin{pmatrix} 280 \\ 120 \end{pmatrix}$

(v) $\begin{pmatrix} 120 \\ 280 \end{pmatrix}$ (vi) $\begin{pmatrix} 280 \\ 120 \end{pmatrix}$ (vii) $\begin{pmatrix} 50 \\ 0 \end{pmatrix}$ (viii) $\begin{pmatrix} 400 \\ 0 \end{pmatrix}$

d) (2 points) Suppose W is a subspace of $\mathbf{R}^n$ and B is the matrix for orthogonal projection onto W. Which of the following must be true? Clearly circle all that apply.

(i) The eigenvalues of B are $\lambda = -1$ and $\lambda = 1$.

$\boxed{\text{(ii)}}$ $B^3 = B$.

Solutions to Problem 5.

a) (i) True: the sum total of algebraic multiplicities of the eigenvalues is 2 since A is 2×2, so if $\lambda = -5$ has algebraic multiplicity 2 then there cannot be any additional eiegenvalues.

(ii) True: if $\lambda = 1 - 4i$ is an eigenvalue, then so is $\lambda = 1 + 4i$, and a 2×2 matrix cannot have more than two eigenvalues, so no eigenvalues of A are real.

(iii) False, for example $A = \begin{pmatrix} 1 & 0.5 \\ 0 & 0.5 \end{pmatrix}$ is stochastic but has eigenvalues $\lambda = 1$ and $\lambda = 0.5$.

b) (i) Not true: the eigenvalues are -1 and 1.

(ii) True: A is 2×2 with the two distinct real eigenvalues -1 and 1, therefore A is diagonalizable.

(iii) True: the (-1)-eigenspace of A is the line through the origin that is perpendicular to $y = 3x$. This is the line $y = -\frac{1}{3}x$, which contains $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Therefore,

$$A\begin{pmatrix} 3 \\ -1 \end{pmatrix} = -\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

c) We are given a positive stochastic matrix and imlicitly told that its 1-eigenspace is spanned by $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$, so its steady-state vector is

$$w = \frac{1}{1+7}\begin{pmatrix} 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 1/8 \\ 7/8 \end{pmatrix}.$$

Therefore, $A^n\left(400\right) \to 400\begin{pmatrix} 1/8 \\ 7/8 \end{pmatrix} = \begin{pmatrix} 50 \\ 350 \end{pmatrix}$.

d) (i) False: orthogonal projection matrices never have $\lambda = -1$ as an eigenvalue.

(ii) True: $B^2 = B$ by a property of orthogonal projections, so $B^3 = B^2B = BB = B$.

Problem 6.

Solutions are on the next page.

a) (2 points) Find the value of c so that the vectors $\begin{pmatrix} 1 \\ -2 \\ c \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ c \\ 0 \\ -3 \end{pmatrix}$ are orthogonal.

$\boxed{c = -3}$

b) (5 points) Suppose W is a subspace of $\mathbf{R}^3$ and x is a vector so that

$$x_W = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad x_{W^\perp} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}.$$

(i) What is the distance from x to W? Clearly circle your answer.

$\boxed{\sqrt{11}}$ $\quad \sqrt{3} \quad \sqrt{30} \quad \sqrt{6} \quad \sqrt{41} \quad 11 \quad 3 \quad 30 \quad 41$

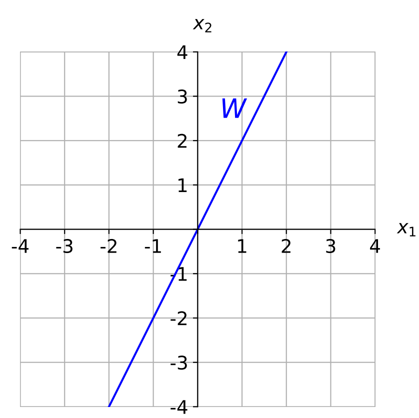
(ii) What is the closest vector to x in W? Clearly circle your answer.

$$\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \quad \begin{pmatrix} -6 \\ 1 \\ 2 \end{pmatrix} \quad \boxed{\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}} \quad \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

(iii) Which **one** of the following **could** be $W^\perp$? Clearly circle your answer.

$$\text{Nul}\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \quad \text{Row}\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \quad \text{Nul}\begin{pmatrix} -1 & 3 & 1 \end{pmatrix} \quad \boxed{\text{Col}\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}} \quad \text{Row}\begin{pmatrix} 5 & 2 & -1 \end{pmatrix}$$

c) (3 points) Let W be the line graphed below, and let $x = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. On the graph below, very carefully **draw** and **label** x, x_W, and $x_{W^\perp}$.



Solutions to Problem 6.

a) We set the dot product of the two vectors equal to 0:

$$0 = 1(6) - 2(c) + c(0) - 12 = -2c - 6,$$

so $c = -3$.

b) (i) The distance from x to W is $||x_{W^\perp}||$:

$$||x_{W^\perp}|| = \sqrt{(-1)^2 + 3^2 + 1^2} = \sqrt{11}.$$

(ii) The closest vector to x in W is x_W, which is $\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$.

(iii) $\mathrm{Nul}\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$ and $\mathrm{Row}\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ are subspaces of $\mathbf{R}^1$ (not $\mathbf{R}^3$), so they are wrong. Also, $\mathrm{Nul}\begin{pmatrix} -1 & 3 & 1 \end{pmatrix}$ and $\mathrm{Row}\begin{pmatrix} 5 & 2 & -1 \end{pmatrix}$ cannot be $W^\perp$ because they do not contain the vector $\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ which we have been told is in $W^\perp$.

We have eliminated every possible answer except $\mathrm{Col}\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$. It could be $W^\perp$, since it contains the vector $\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ which we know is in $W^\perp$.

One possibility is that $W = \mathrm{Span}\left\{\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$ and $W^\perp = \mathrm{Span}\left\{\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}\right\}$. In this setup, we see that for $x = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}$, we get: $\quad x_W = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}, \quad x_{W^\perp} = x - x_W = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$.

c) One way to do this problem with no algebra is just to draw x and then complete a right triangle since x is the sum of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in W and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ in $W^\perp$. Alternatively, with $W = \mathrm{Span}\{u\}$ for $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, we could compute

$$x_W = \frac{1}{u \cdot u} u u^T x = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad x_{W^\perp} = x - x_W = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Many students drew lines instead of vectors, which is badly incorrect. The entire point of the problem was to draw the three vectors, not their spans. Many students also did not label (or incorrectly labeled) the vectors involved.

Problem 7.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

For this problem, let $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 3 & -15 & 13 \end{pmatrix}$.

a) (2 pts) Write the eigenvalues of A. You do not need to show your work on this part.

Fill in the blank: the eigenvalues are $\lambda = 4$ and $\lambda = 13$.

b) (6 points) For each eigenvalue of A, find a basis for the corresponding eigenspace.

Solution:

$$\lambda = 4: \ \left(A - 4I \mid 0\right) = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & -15 & 9 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 3 & -15 & 9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & -5 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so $x_1 = 5x_2 - 3x_3$ where x_2 and x_3 are free.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

4-eigenspace basis: $\left\{ \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

$$\lambda = 13: \ \left(A - 13I \mid 0\right) = \left(\begin{array}{ccc|c} -9 & 0 & 0 & 0 \\ 0 & -9 & 0 & 0 \\ 3 & -15 & 0 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

so $x_1 = 0$, $x_2 = 0$, and x_3 is free. The 13-eigenspace has basis $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

c) (2 points) The matrix A is diagonalizable. Write a 3×3 matrix C and a 3×3 diagonal matrix D so that $A = CDC^{-1}$. Enter your answer below.

We form C using linearly independent eigenvectors and form D using the eigenvalues written **in the corresponding order**. Many answers are possible. For example,

$$C = \begin{pmatrix} 5 & -3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 13 \end{pmatrix}$$

or

$$C = \begin{pmatrix} 0 & 5 & -3 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 13 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Problem 8.

a) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation corresponding to reflection over the line $y = x$. Find the standard matrix A for T, so $T(v) = Av$. Enter your answer below.

$$A = \left(T\begin{pmatrix}1\\0\end{pmatrix} \quad T\begin{pmatrix}0\\1\end{pmatrix} \right) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

b) Let $S : \mathbf{R}^3 \to \mathbf{R}^2$ be the transformation defined by $S(x, y, z) = (2x - y, y + 4z)$. Find the standard matrix B for S, so $S(v) = Bv$. Enter your answer below.

$$B = \left(S\begin{pmatrix}1\\0\\0\end{pmatrix} \quad S\begin{pmatrix}0\\1\\0\end{pmatrix} \quad S\begin{pmatrix}0\\0\\1\end{pmatrix} \right) = \begin{pmatrix}2 & -1 & 0\\ 0 & 1 & 4\end{pmatrix}$$

c) Which of the following expressions are possible to calculate? Clearly circle all that apply. You do not need to show your work on this part.

$$T\begin{pmatrix}3\\-5\\8\end{pmatrix} \qquad \boxed{S\begin{pmatrix}6\\2\\-1\end{pmatrix}} \qquad \boxed{(T\circ S)\begin{pmatrix}1\\0\\1\end{pmatrix}} \qquad (S\circ T)\begin{pmatrix}1\\0\\1\end{pmatrix}$$

d) Which one of the following compositions makes sense? Circle the **one** correct answer. You do not need to show your work on this part.

$$\boxed{T \circ S} \qquad S \circ T$$

e) Find the standard matrix C for the transformation you circled in part (d). Enter your answer below.

$$C = \begin{pmatrix}0 & 1 & 4\\ 2 & -1 & 0\end{pmatrix}$$

Solution.

Parts (a) and (b) are done above. For part (c), we see $T\begin{pmatrix}3\\-5\\8\end{pmatrix}$ is undefined, and $S \circ T$ is undefined because the domain of S is $\mathbf{R}^3$. On the other hand, $S\begin{pmatrix}6\\2\\-1\end{pmatrix}$ and $(T\circ S)\begin{pmatrix}1\\0\\1\end{pmatrix}$ make sense and we could compute them if we wished.

For parts (d) and (e), we see $T \circ S$ makes sense because the domain of T is the codomain of S, and the matrix C for $T \circ S$ is equal to AB:

$$C = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}2 & -1 & 0\\ 0 & 1 & 4\end{pmatrix} = \begin{pmatrix}0 & 1 & 4\\ 2 & -1 & 0\end{pmatrix}.$$

Problem 9.

Free response. Show your work! A correct answer without sufficient work may receive little or no credit. Parts (a) and (b) are unrelated.

a) Let $W = \text{Span}\left\{\begin{pmatrix}-1\\6\\-3\end{pmatrix}\right\}$. Find a basis for $W^\perp$.

Solution: $W = \text{Col}(A)$ for the matrix $A = \begin{pmatrix}-1\\6\\-3\end{pmatrix}$, so

$$W^\perp = (\text{Col } A)^\perp = \text{Nul}(A^T) = \text{Nul}\begin{pmatrix}-1 & 6 & -3\end{pmatrix}.$$

This gives $-x_1+6x_2-3x_3 = 0$, so $x_1 = 6x_2-3x_3$ with x_2 and x_3 free. The parametric vector form is

$$\begin{pmatrix}6x_2-3x_3\\x_2\\x_3\end{pmatrix} = x_2\begin{pmatrix}6\\1\\0\end{pmatrix} + x_3\begin{pmatrix}-3\\0\\1\end{pmatrix},$$

so a basis for $W^\perp$ is $\left\{\begin{pmatrix}6\\1\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\end{pmatrix}\right\}$.

b) Let $W = \text{Span}\left\{\begin{pmatrix}1\\-4\end{pmatrix}\right\}$ and let $x = \begin{pmatrix}1\\-1\end{pmatrix}$.

Find x_W (the orthogonal projection of x onto W) and $x_{W^\perp}$. Enter your answers below. **Simplify all fractions in your answer as much as possible.**

$x_W = \begin{pmatrix}5/17\\-20/17\end{pmatrix}$ $\qquad$ $x_{W^\perp} = \begin{pmatrix}12/17\\3/17\end{pmatrix}$.

Solution: The matrix B for orthogonal projection onto W is

$$B = \frac{1}{u\cdot u}uu^T = \frac{1}{1^2+(-4)^2}\begin{pmatrix}1\\-4\end{pmatrix}\begin{pmatrix}1 & -4\end{pmatrix} = \frac{1}{17}\begin{pmatrix}1 & -4\\-4 & 16\end{pmatrix}.$$

Now, $x_W = Bx = \frac{1}{17}\begin{pmatrix}1 & -4\\-4 & 16\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{17}\begin{pmatrix}5\\-20\end{pmatrix} = \begin{pmatrix}5/17\\-20/17\end{pmatrix}$ and

$$x_{W^\perp} = x - x_W = \begin{pmatrix}1\\-1\end{pmatrix} - \begin{pmatrix}5/17\\-20/17\end{pmatrix} = \begin{pmatrix}12/17\\3/17\end{pmatrix}.$$

Problem 10.

Free response. Show your work!
Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(0, 5), \qquad (2, -5), \qquad (4, -3).$$

Enter your answer below:

$$y = \_\_\_\_\_\_ x \;+\; \_\_\_\_\_\_.$$

You **must** show appropriate work using least squares. If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

No line goes through all three points. The corresponding (inconsistent) system is

$$\begin{aligned} 5 &= M(0) + B \\ -5 &= M(2) + B \\ -3 &= M(4) + B \end{aligned}$$

and the corresponding matrix equation is $Ax = b$ where $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 5 \\ -5 \\ -3 \end{pmatrix}$.

We solve $A^T A\widehat{x} = A^T b$.

$$A^T A = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 20 & 6 \\ 6 & 3 \end{pmatrix}, \qquad A^T b = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ -5 \\ -3 \end{pmatrix} = \begin{pmatrix} -22 \\ -3 \end{pmatrix}.$$

$$\left(A^T A \mid A^T b \right) = \left(\begin{array}{cc|c} 20 & 6 & -22 \\ 6 & 3 & -3 \end{array}\right) \xrightarrow[\text{then } R_1 = R_1/3]{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 2 & 1 & -1 \\ 20 & 6 & -22 \end{array}\right) \xrightarrow{R_2 = R_2 - 10R_1} \left(\begin{array}{cc|c} 2 & 1 & -1 \\ 0 & -4 & -12 \end{array}\right)$$

$$\xrightarrow[R_1 = R_1/2]{R_2 = -R_2/4} \left(\begin{array}{cc|c} 1 & 1/2 & -1/2 \\ 0 & 1 & 3 \end{array}\right) \xrightarrow{R_1 = R_1 - (1/2)R_2} \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \end{array}\right).$$

Thus $\widehat{x} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$. The line is

$$y = -2x + 3.$$

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If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.