MATH 1553, SPRING 2024

SAMPLE MIDTERM 3A: COVERS 4.1 THROUGH 5.6

Name		**GT ID**	

Circle your lecture below.

Jankowski (A and HP, 8:25-9:15 AM) Jankowski (G, 12:30-1:20 PM)

Hausmann (I, 2:00-2:50 PM) Sanchez-Vargas (M, 3:30-4:20 PM)

Athanasouli (N and PNA, 5:00-5:50 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. We recommend completing the practice exam in 75 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to section §4.1 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to §§4.1 through 5.6.

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Problem 1.

For each statement, answer TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

a) If A is a 4×4 matrix, then $\det(-A) = -\det(A)$.

TRUE FALSE

b) Suppose A is an $n \times n$ matrix and B is its RREF. If $\det(B) = 0$, then $\det(A) = 0$.

TRUE FALSE

c) If u and v are eigenvectors of an $n \times n$ matrix A, then $u + v$ must also be an eigenvector of A.

TRUE FALSE

d) Suppose A is a 2×2 matrix. If $A\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}6\\3\end{pmatrix}$, then $A - 3I$ is not invertible.

TRUE FALSE

e) Suppose A is a 6×6 matrix with exactly two eigenvalues $\lambda = 2$ and $\lambda = -5$. If $\dim(\mathrm{Nul}(A - 2I)) = 5$, then A must be diagonalizable.

TRUE FALSE

Solution.

a) False: if A is 4×4 then $\det(-A) = (-1)^4 \det(A) = \det(A)$.

b) True: some row-reduction operations change the value of the determinant, but they *never* change a nonzero determinant to a zero determinant (or vice versa).

Another way to do this is via an invertibility argument. If $\det(A) = 0$ then A is not invertible and so its RREF B has fewer than n pivots and is also therefore not invertible, so $\det(B) = 0$. On the other hand, if $\det(A) \neq 0$ then A is invertible, and the RREF of A is I and $\det(I) = 1$.

c) False. If u and v belong to different eigenspaces of A, then $u + v$ will not be an eigenvector of A. For example, if $A = \begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}$ then $u = \begin{pmatrix}1\\0\end{pmatrix}$ is an eigenvector for $\lambda = 1$ and $v = \begin{pmatrix}0\\1\end{pmatrix}$ is an eigenvector for $\lambda = 2$, but $u + v$ is not an eigenvector since $A(u+v) = A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$. This problem is #1g in the 5.1 Supplement.

d) True: $A\begin{pmatrix}2\\1\end{pmatrix} = 3\begin{pmatrix}2\\1\end{pmatrix}$ so $\lambda = 3$ is an eigenvalue of A, therefore $A - 3I$ is not invertible as a fundamental consequence.

e) True: $\dim(\mathrm{Nul}(A + 5I)) \geq 1$ since $\lambda = -5$ is an eigenvalue, and we already know $\dim(\mathrm{Nul}(A - 2I)) = 5$. This means the sum of dimensions of eigenspaces of A is at least 6, thus exactly 6 (since A is 6×6) and A is diagonalizable.

Problem 2.

Short answer and multiple choice. You do not need to show your work, and there is no partial credit.

a) Let A and B be 3×3 matrices with $\det(A)=4$ and $\det(B)=-2$. Find the determinant of $(A^T B^{-1})^2$.

(i) -4 (ii) 2 (iii) -2 (iv) $1/4$

(v) 4 (vi) $1/16$ (vii) 16 (viii) none of these

b) Match each of the following linear transformations $\mathbf{R}^2 \to \mathbf{R}^2$ with the eigenvalues of its standard matrix. For each roman numeral (i) through (iv), write the corresponding set of eigenvalues (one of the choices (1) through (5)). It is possible that the same answer choice will be used more than once.

(i) Rotation countercl. by $\pi/3$ radians
(ii) Rotation clockwise by π radians
(iii) Reflection across the line $y=x$
(iv) Projection onto the y-axis

(1) $\lambda=0$ and $\lambda=1$
(2) $\lambda=1$ and $\lambda=-1$
(3) $\lambda=1$ only
(4) $\lambda=-1$ only
(5) No real eigenvalues

Write your answer for (i) here: ________

Write your answer for (ii) here: ________

Write your answer for (iii) here: ________

Write your answer for (iv) here: ________

c) Let $A=\begin{pmatrix}4&1\\2&3\end{pmatrix}\begin{pmatrix}-1/2&0\\0&1\end{pmatrix}\begin{pmatrix}4&1\\2&3\end{pmatrix}^{-1}$. Which of the following statements are true? Select all that apply.

(i) $A^n\begin{pmatrix}4\\2\end{pmatrix}$ approaches the zero vector as $n\to\infty$.

(ii) $A\left(\begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}1\\3\end{pmatrix}\right)=\begin{pmatrix}-1\\2\end{pmatrix}$.

(iii) Repeated multiplication by A pushes vectors towards the $(-1/2)$-eigenspace.

(iv) A is invertible.

Solution.

a) $\det\left((A^T B^{-1})^2\right) = \left(4 \cdot \frac{1}{-2}\right)^2 = (-2)^2 = 4$. For practice with similar problems, see the 4.1-5.1 Worksheet.

b) All parts (i)-(iv) deal with eigenvalues of rotations, reflections, and projections.

(i) The answer is (5) because rotation counterclockwise by $\pi/3$ has no real eigenvalues.

(ii) The answer is (4) because rotation clockwise by π radians is just multiplication by $-I$.

(iii) The answer is (2) because reflection across $y = x$ fixes vectors along the line $y = x$ and flips vectors on the perpendicular line $y = -x$.

(iv) The answer is (1) because this projection destroys vectors on the x-axis and fixes vectors on the y-axis.

c) (i) True: $A^n \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \left(-\frac{1}{2}\right)^n \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ which shrinks towards the zero vector as $n \to \infty$.

(ii) True: $A\left(\begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right) = -\frac{1}{2}\begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

(iii) False: repeated multiplication by A pushes vectors towards the 1-eigenspace.

(iv) True: A is 2×2 with eigenvalues $-1/2$ and 1, so 0 is not an eigenvalue of A. Alternatively, we could see A is invertible by observing that A is the product of three invertible matrices and is thus invertible:

$$A^{-1} = (CDC^{-1})^{-1} = (C^{-1})^{-1} D^{-1} C^{-1} = CD^{-1}C^{-1}.$$

Problem 3.

Short answer and multiple choice. Show your work on part (a). You do not need to show your work on parts (b) and (c), and there is no partial credit on (b) and (c).

a) (3 points) Find the area of the triangle with vertices $(0,0)$, $(1,4)$, and $(5,2)$. Show your work on this part of the problem.

b) (4 points) Suppose A is a square matrix and its characteristic polynomial is

$$\det(A-\lambda I) = -\lambda^3 - 2\lambda^2 + 1.$$

Which of the following statements must be true? Circle all that apply.

(i) A is a 3×3 matrix.

(ii) $\det(A) = -1$.

(iii) The matrix transformation $T(v) = Av$ is onto.

(iv) $A - I$ is invertible.

c) (3 points) Suppose A is a 2×2 positive stochastic matrix. Which of the following statements are true? Circle all that apply.

(i) The equation $(A-I)x = 0$ has exactly one solution.

(ii) The 1-eigenspace of A is a line.

(iii) All of the eigenvalues of A are real.

Solution.

a) Basically from the Determinants I Webwork:

$$\frac{1}{2}\left|\det\begin{pmatrix}1 & 5\\ 4 & 2\end{pmatrix}\right| = \frac{1}{2}|2-20| = \frac{1}{2}(18) = 9.$$

b) (i) True: the characteristic polynomial has degree 3, so A is 3×3.

(ii) False: $\det(A) = \det(A-0I) = -0-0+1 = 1$. Same idea as #6 from the 5.2 Webwork.

(iii) True: $\det(A) \neq 0$, so A is invertible and thus T is one-to-one and onto. Similar to the Det. II Webwork #8.

(iv) True: $\det(A-I) = -1-2+1 = -2 \neq 0$.

c) (i) False: $\lambda = 1$ is an eigenvalue, so $A-I$ is not invertible and therefore $(A-I)x = 0$ has infinitely many solutions. This is a fundamental consequence of the concept of eigenvalue.

(ii) True, by the Perron-Frobenius Theorem.

(iii) True: 1 is an eigenvalue, and if A had a (non-real) complex eigenvalue λ then its complex conjugate $\overline{\lambda}$ would also be an eigenvalue of A, giving A three eigenvalues. This is impossible, since any 2×2 matrix has at most two different eigenvalues.

Problem 4.

Short answer and multiple choice. You do not need to show your work on this page, and there is no partial credit. Parts (a) through (d) are unrelated.

a) (2 points) Write the value of c so that $\lambda = 1$ is an eigenvalue of the matrix $\begin{pmatrix} 2 & 4 \\ 3 & c \end{pmatrix}$.

Fill-in the blank: $c =$ ________.

b) (3 points) Let A be a 4×4 matrix whose entries are real numbers. Which polynomials below are possible for the characteristic polynomial of A? Select all that apply.

(i) $(\lambda-1)^2(\lambda-2)^2$

(ii) $(\lambda-i)^2(\lambda-1)^2$

(iii) $(\lambda-1)^3(\lambda+1)(\lambda-2)$

c) (3 points) In each case, determine whether the matrix is diagonalizable. Clearly circle your answers.

(i) $A = \begin{pmatrix} 3 & 4 \\ 0 & 3 \end{pmatrix}$ Diagonalizable Not Diagonalizable

(ii) $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ Diagonalizable Not Diagonalizable

(iii) $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ Diagonalizable Not Diagonalizable

d) (2 points) Consider the positive stochastic matrix $A = \begin{pmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{pmatrix}$. The 1-eigenspace of A is spanned by $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$. What vector does $A^n\begin{pmatrix} 17 \\ 4 \end{pmatrix}$ approach as n gets very large?

(i) $\begin{pmatrix} 9 \\ 12 \end{pmatrix}$ (ii) $\begin{pmatrix} 17 \\ 4 \end{pmatrix}$ (iii) $\begin{pmatrix} 51 \\ 16 \end{pmatrix}$ (iv) $\begin{pmatrix} 9/21 \\ 12/21 \end{pmatrix}$ (v) $\begin{pmatrix} 51/67 \\ 16/67 \end{pmatrix}$ (vi) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Solution.

a) $A-I = \begin{pmatrix} 1 & 4 \\ 3 & c-1 \end{pmatrix}$, which is not invertible precisely when $(c-1)-12 = 0$, so $c = 13$.
This problem is a slight modification of #1 from the 5.1 Webwork.

b) (i) Possible.
(ii) Impossible. If i is an eigenvalue, then $-i$ must be an eigenvalue, but the only eigenvalues here are i and 1.
(iii) Impossible. This polynomial has degree 5, but a 4×4 matrix's characteristic polynomial must have degree 4.

c) (i) Not diagonalizable. Its only eigenvalue is $\lambda = 3$, but the 3-eigenspace is only 1-dimensional.

(ii) Diagonalizable. The matrix is 3×3 with 3 distinct real eigenvalues.

(iii) Diagonalizable. The characteristic polynomial is $\lambda^2 - 2\lambda$, so A is 2×2 with 2 different real eigenvalues 0 and 2.

d) (i) is the answer. The steady-state vector is $\begin{pmatrix} 3/7 \\ 4/7 \end{pmatrix}$. By Perron-Frobenius,

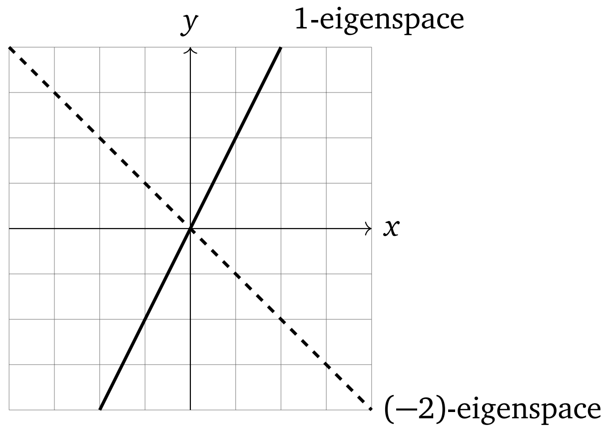
$$A^n \begin{pmatrix} 17 \\ 4 \end{pmatrix} \to (17+4) \begin{pmatrix} 3/7 \\ 4/7 \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \end{pmatrix}.$$

It is also possible to do this problem by the process of elimination. Our answer must be a nonzero scalar multiple of $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, and there are only two choices that satisfy this (namely (i) and (iv)). However, the sum of entries in answer (iv) is nowhere close to 21, so (i) must be correct.

Problem 5.

Free response! Show your work. Parts (a) and (b) are unrelated.

a) (4 points) Let A be a 2×2 matrix whose 1-eigenspace is the **solid** line below and whose (-2)-eigenspace is the **dashed** line below.



Find $A\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

b) (6 points) Let $A = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$. Find the eigenvalues of A. For the eigenvalue whose imaginary part is *positive*, find one corresponding eigenvector.

Solution.

a) By the picture, the 1-eigenspace is spanned by $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and the (-2)-eigenspace is spanned by $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Note $\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so

$$A\begin{pmatrix} 0 \\ 3 \end{pmatrix} = A\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} - A\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - (-2)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

b) This is a standard problem with complex eigenvalues. The characteristic polynomial is

$$\det(A - \lambda I) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 4\lambda + 5.$$

Therefore, the eigenvalues are

$$\lambda = \frac{4 \pm \sqrt{4^2 - 5(4)}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i.$$

For $\lambda = 2 + i$, we use the 2×2 trick: If the first row of $A - \lambda I$ is $\begin{pmatrix} a & b \end{pmatrix}$ (where a and b are not both zero), then $\begin{pmatrix} -b \\ a \end{pmatrix}$ is an eigenvector of A.

$$\left(A - (2+i) \mid 0\right) = \left(\begin{array}{cc|c} 1-i & -2 & 0 \\ (*) & (*) & 0 \end{array}\right),$$

so $v = \begin{pmatrix} 2 \\ 1-i \end{pmatrix}$ is an eigenvector. An equivalent answer is $\begin{pmatrix} -2 \\ -1+i \end{pmatrix}$ which is just $-v$. Other answers are also possible, for example $\begin{pmatrix} 1+i \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1/2-i/2 \end{pmatrix}$.

Problem 6.

Free response. Show your work!

Consider $A = \begin{pmatrix} 1 & 5 & 10 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$.

a) (2 points) Write the eigenvalues of the matrix A.

b) (5 points) For each eigenvalue, find a basis for the corresponding eigenspace.

c) (3 points) Is A diagonalizable? If so, write an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$.

Solution.

This is a quintessential eigenvalues / eigenvectors / diagonalization problem.

a) Here A is upper-triangular, so its eigenvalues are the diagonal entries $\lambda = 1$ and $\lambda = -4$.

b) For $\lambda = 1$:

$$\big(A - I \mid 0\big) = \left(\begin{array}{ccc|c} 0 & 5 & 10 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This gives x_1 free, $x_2 = 0$, and $x_3 = 0$, so $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ is a basis for the 1-eigenspace.

For $\lambda = -4$:

$$\big(A + 4I \mid 0\big) = \left(\begin{array}{ccc|c} 5 & 5 & 10 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus $x_1 = -x_2 - 2x_3$ and x_2 and x_3 are free. In parametric vector form,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - 2x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for the (-4)-eigenspace.

c) Yes, A is 3×3 with 3 linearly independent eigenvectors, so A is diagonalizable. One diagonalization is

$$C = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

Another possibility, with eigenvectors and eigenvalues written in a different (but still matching!) order, is

$$C = \begin{pmatrix} -1 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

Problem 7.

Free response. Show your work! Parts (a) and (b) are unrelated.

a) Suppose

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = 18.$$

Find the determinant of the matrix B given below.

$$B = \begin{pmatrix} 3a+2b & c & b & \pi \\ 3d+2e & f & e & \ln 2 \\ 3g+2h & k & h & e \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

b) Amara's coffee shop competes with Ben's cafe for customers. Currently in 2022, Amara has 210 customers and Ben has 140 customers.

Each year:

- 70% of Amara's customers stay with Amara, while 30% of Amara's customers switch to Ben.
- 60% of Ben's customers stay with Ben, while 40% of Ben's customers switch to Amara.

Answer the following questions.

(i) (4 points) Write a positive stochastic matrix A and a vector x so that Ax is the vector that gives the number of customers for Amara and Ben (in that order) in 2023. Do not carry out the multiplication! Just write A and x.

(ii) (3 points) Find the steady-state vector w for the matrix A.

Solution.

a) We use the cofactor expansion along the 4th row.

$$\det(B) = -2(-1)^{4+4}\det\begin{pmatrix} 3a+2b & c & b \\ 3d+2e & f & e \\ 3g+2h & k & h \end{pmatrix} = -2\det\begin{pmatrix} 3a+2b & c & b \\ 3d+2e & f & e \\ 3g+2h & k & h \end{pmatrix}.$$

$$\text{(take transpose)} = -2\det\begin{pmatrix} 3a+2b & 3d+2e & 3g+2h \\ c & f & k \\ b & e & h \end{pmatrix}$$

$$\text{(do row replacement)} = -2\det\begin{pmatrix} 3a & 3d & 3g \\ c & f & k \\ b & e & h \end{pmatrix}$$

$$\text{(take out row multiple of 3)} = -6\det\begin{pmatrix} a & d & g \\ c & f & k \\ b & e & h \end{pmatrix}$$

$$\text{(swap rows 2 and 3)} = 6\det\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & k \end{pmatrix}$$

$$\text{(take transpose)} = 6\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = 6(18) = 108.$$

b) (i) $A = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}$ and $x = \begin{pmatrix} 210 \\ 140 \end{pmatrix}$.

(ii) $\left(A - I \mid 0\right) = \left(\begin{array}{cc|c} -0.3 & 0.4 & 0 \\ 0.3 & -0.4 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -4/3 & 0 \\ 0 & 0 & 0 \end{array}\right)$. Therefore, the 1-eigenspace is spanned by $\begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$ (or equivalently by $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$), so the steady-state vector is

$$w = \frac{1}{4/3+1}\begin{pmatrix} 4/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4/7 \\ 3/7 \end{pmatrix}.$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.