Discrete Mathematics Comprehensive Exam
Spring 2020

Student Number: ☐

Instructions: Complete 5 of the 6 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let G be a connected triangle-free graph that does not contain $K_{1,t}$ as an induced subgraph (where t is a positive integer). Show $\chi(G) \leq t$ and determine the values of t for which we could have $\chi(G) = t$.

2. Let $\mathcal{B}$ be a basis of the cycle space of K_5. Then some edge of K_5 is contained in at least three members of $\mathcal{B}$.

3. Let G be a 4-edge-connected 5-regular graph. Show that G contains a perfect matching.

4. For a bipartite graph $G = (X \cup Y, E)$, with X and Y being the two (disjoint) parts of the vertex set, we say $Y' \subseteq Y$ *covers* X, if every $x \in X$ is adjacent to some $y \in Y'$.

   Suppose $d(x) \geq a$, $\forall x \in X$ and $d(y) \leq b$, $\forall y \in Y$. Then show that there exists a Y' that i) covers X and ii) of size at most $(|Y|/a)[1 + \ln b]$.

5. Let $A_1, A_2, \ldots, A_m$ be r-element sets and $B_1, B_2, \ldots, B_m$ be s-element sets. Suppose $A_i \cap B_i = \phi$ for each i and for each pair $i \neq j$, either $A_i \cap B_j \neq \phi$ or $A_j \cap B_i \neq \phi$. Prove that $m \leq (r+s)^{r+s}/(r^r s^s)$, using the following steps.

   i) Set $S := A_1 \cup A_2 \cup \cdots \cup A_m \cup B_1 \cup B_2 \cup \cdots \cup B_m$. For each element of S, independently assign it to S_a with probability $r/(r+s)$ and assign it to S_b otherwise. Let E_i be the event that $A_i \subset S_a$ and $B_i \subset S_b$.

   Compute the expected number of events E_i that occur, in terms of m, r and s.

   ii) Can more than one event E_i occur?

6. Let $G = (V, E)$ be a graph with maximum degree d and let $V = V_1 \cup V_2 \cup \cdots \cup V_r$ be a partition of the vertex set into r pairwise disjoint sets. Suppose each V_i is of cardinality at least $6d$, then show that there is an independent set $I \subset V$ of vertices that contains a vertex each from V_i.