PROBABILITY COMPREHENSIVE EXAM SPRING 2021

Student Number: ☐

Instructions: Complete 5 of the 9 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8 9

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

Problem 1. *Construct an example of a non-negative integer valued random variable X such that $\mathbb{E}(X!)$ is finite but* $\operatorname{esssup}(X) := \sup\big\{t \geq 0 :\ \mathbb{P}\{X \geq t\} > 0\big\} = \infty$.

Problem 2. *Let $0 < p < +\infty$, and let X and Y be two independent random variables such that $\mathbb{E}|X+Y|^p < +\infty$. Is it then true or false that both $\mathbb{E}|X|^p < +\infty$ and $\mathbb{E}|Y|^p < +\infty$? What about if X and Y are no longer independent?*

Problem 3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and assume that it admits a sequence $(b_i)_{i=1}^{\infty}$ of independent identically distributed (i.i.d.) Bernoulli random variables with parameter $1/2$. Prove that the set Ω is uncountable.*

Problem 4. *Let $(X_n)_{n=1}^{\infty}$ be a sequence of mutually independent random variables, where for each $n \geq 1$, X_n has density*

$$\rho_n(t) := \begin{cases} \alpha_n\Big(\frac{1}{\sqrt{2\pi}}\exp(-t^2/2) + \frac{1}{n^{4.1}}\Big), & \textit{if } |t| \leq n^3; \\ \frac{\alpha_n}{\sqrt{2\pi}}\exp(-t^2/2), & \textit{otherwise}, \end{cases}$$

where $\alpha_n := \big(1 + 2n^{-1.1}\big)^{-1}$. Prove that the sequence rescaled partial sums

$$\frac{1}{\sqrt{n}}\big(X_1 + X_2 + \cdots + X_n\big), \quad n \geq 1,$$

converges in distribution to a standard normal variable.

Problem 5. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $\lim\limits_{n\to\infty}(a_1^2 + \cdots + a_n^2) = \infty$. For each n, let X_n be uniform on $[0, a_n]$, and assume that $(X_n)_{n=1}^{\infty}$ are mutually independent. Show that the sequence*

$$\mathbb{E}\sin\big(X_1 + \cdots + X_n\big), \quad n \geq 1,$$

converges to zero.

Problem 6. *Let the random variable X be uniformly distributed on the set*

$$[0,1) \setminus \Big(\bigcup_{i=1}^{\infty}[1 - 2^{-i}, 1 - 2^{-i} + 2^{-i-2})\Big).$$

Compute the mean of X.

Problem 7. *Let $(X_n)_{n\geq 1}$ be a sequence of iid random variables with $\mathbb{E}X_1 = 0$, $0 < \mathbb{E}X_1^2 < +\infty$, and let $(a_n)_{n\geq 1}$ be a sequence of reals. Show that $\sum_{n=1}^{+\infty} a_n X_n$ converges almost surely if and only if $\sum_{n=1}^{+\infty} a_n^2 < +\infty$.*

Problem 8. *On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $(X_n)_{n\geq 1}$ be a sequence of independent Poisson random variables each with parameter λ_n and such that $\sum_{n=1}^{+\infty} \lambda_n = +\infty$. Let $S_n = \sum_{k=1}^{n} X_k$. Does $S_n/\mathbb{E}S_n$ converges in distribution as $n \to +\infty$? If yes, towards which limit? Does the convergence also hold $L^2(\Omega, \mathcal{F}, \mathbb{P})$?*

Problem 9. *Let $G = (V, E)$ an an infinite binary rooted tree with root $v_r \in V$ (we recall that an infinite binary rooted tree is an infinite undirected simple connected graph with no cycles, such that the degree of the root is two, and the degree of any vertex apart from the root is three). Let $(X_n)_{n=0}^{\infty}$ be the simple random walk on G starting at a vertex $X_0 = v_0$, that is, $(X_n)_{n=0}^{\infty}$ is a Markov chain on the vertices of G with transition probabilities*

$$\mathbb{P}\big\{X_{n+1} = w \mid X_n = v\big\} = \begin{cases} \frac{1}{\deg(v)}, & \text{if } w \text{ is adjacent to } v; \\ 0, & \text{otherwise}. \end{cases}$$

Assume that v_0 is at distance $k \geq 1$ from the root (that is, the length of the shortest path connecting v_0 and v_r is k). Compute the probability that the walk $(X_n)_{n=0}^{\infty}$ visits the root at least once (the answer should be a function of k).