Algebra Comprehensive Exam
Spring 2021

1. Let $\mathbf{F}$ be a finite field and let M be an invertible $n \times n$ matrix with entries in $\mathbf{F}$. Prove that $M^m - I_n$ is not invertible for some integer $m \geq 1$. (I_n denotes the $n \times n$ identity matrix.)

2. Construct a nonabelian group of order $150 = 2 \cdot 3 \cdot 5^2$ whose Sylow 5-subgroup is not cyclic.

3. Let H be a subgroup and N be a normal subgroup of a group G. Suppose $[G : H]$ and $|N|$ are finite and are relatively prime to each other. For each of the following statements, either prove or give a counterexample.
   (a) $N \subseteq H$.
   (b) If G is finite and N is nontrivial, then $H \subseteq N$.

4. Let G be an abelian group with generators a, b, c and relations
$$2a + 10b + 6c = -4a - 6b - 12c = -2a + 4b - 6c = 0$$
   (a) Find the decomposition of G according to the Fundamental Theorem of finitely generated abelian groups.
   (b) What are cyclic generators corresponding to the components in this decomposition in terms of a, b, c?

5. Let R be a commutative ring with unity. Let I be a nontrivial prime ideal in R. Prove that R/I satisfies the descending chain condition if and only if it is a field.
(The descending chain condition means that any chain of ideals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ stabilizes in R/I.)

6. Let R be a unique factorization domain.
   (a) Show that every ascending chain of principal ideals $I_1 \subseteq I_2 \subseteq J_3 \subseteq \cdots$ in R must stabilize.
   (b) Is the statement still true if we remove the word "principal"? Justify your answer.

7. Let K be the splitting field of $x^{13} - 1$. What are possible degrees of elements in K over $\mathbb{Q}$? Find an element of each possible degree.