Student Number: 

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7 8

Write only on the front side of the solution pages. A student will pass the exam if 3 problems are worked “almost perfectly” and some progress is made on a fourth problem.
1. Let $A$ and $B$ be $n \times n$ matrices over a field $K$ such that $A^2 = A$ and $B^2 = B$. Prove that $A$ and $B$ are similar if and only if they have the same rank.

2. If $A$ and $B$ are normal subgroups of a group $G$ such that $G/A$ and $G/B$ are both abelian, prove that $G/(A \cap B)$ is abelian.

3. Let $R$ be a principal ideal domain, let $M$ be a torsion $R$-module, and let $p$ be a prime in $R$. Prove that if $pm = 0$ for some nonzero $m \in M$, then the annihilator $Ann(M)$ is a subset of the ideal $\langle p \rangle$.
   Recall that $Ann(M) = \{ r \in R \mid rm = 0 \text{ for all } m \in M \}$.

4. Show that in a finite field every element is a sum of two perfect squares. (0 counts as a perfect square.)

5. (a) Show that every prime ideal in a principal domain is maximal.
   (b) Let $R$ be a ring with a unique maximal ideal $M$. Show that an element of $R$ is invertible if and only if it is not in $M$.

6. Let $p$ be a prime number. Find two non-isomorphic groups of order $2p$. Show that, up to isomorphism, there are only two groups of order $2p$.

7. Let $R$ be a commutative ring. A polynomial over $R$ is called primitive if its coefficients generate $R$. If $f, g \in R[x]$, show that $f \cdot g$ is primitive if and only if both $f$ and $g$ are primitive. (hint: consider a maximal ideal containing the coefficients of $f \cdot g$).

8. Let $\zeta$ be a primitive 11-th root of unity. Use the Galois correspondence to determine the degrees of $\alpha = \zeta^3 + \zeta^8 + 6$ and of $\beta = \zeta^2 + \zeta^3$ over $\mathbb{Q}$.