Algebra Comprehensive Exam
Spring 2020

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let A and B be complex matrices such that $A^*A + B^*B = 0$. Prove that $A = 0$ and $B = 0$.

   You may use the Spectral Theorem for Hermitian matrices, but all other statements must be proved if using.

2. Prove that if G is a finite group containing no subgroup of index 2, then any subgroup of index 3 is normal in G. Hint: If H is a subgroup of index 3, consider the action of G on the cosets G/H by left multiplication.

3. How many groups are there of order 44, up to isomorphism? How many of them are abelian?

4. (a) Show that every nonzero prime ideal in a principal ideal domain is a maximal ideal.

   (b) Let R be a commutative ring with identity. If the polynomial $R[x]$ is a principal ideal domain, then show that R is a field.

5. Let A be a commutative ring with identity and let N be the ideal of nilpotent elements. Show the following are equivalent:

   (a) A has exactly one prime ideal.

   (b) Every element of A is either a unit or nilpotent.

   (c) A/N is a field.

   You may use the fact that N coincides with the intersection of all prime ideals in A.

6. Let M be a module over a commutative ring R. Show that $\mathrm{Hom}_R(R, M) \simeq M$ as R-modules.

7. Explicitly construct a Galois extension K over $\mathbb{Q}$ whose Galois group is cyclic of order 8.

8. If $f \in \mathbb{Q}[x]$ is an irreducible polynomial of odd degree with abelian Galois group then all its roots are real.