Algebra Comprehensive Exam
Spring 2020

Student Number: 

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7 8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.
1. Let $A$ and $B$ be complex matrices such that $A^*A + B^*B = 0$. Prove that $A = 0$ and $B = 0$.
   You may use the Spectral Theorem for Hermitian matrices, but all other statements must be proved if using.

2. Prove that if $G$ is a finite group containing no subgroup of index 2, then any subgroup of index 3 is normal in $G$. Hint: If $H$ is a subgroup of index 3, consider the action of $G$ on the cosets $G/H$ by left multiplication.

3. How many groups are there of order 44, up to isomorphism? How many of them are abelian?

4. (a) Show that every nonzero prime ideal in a principal ideal domain is a maximal ideal.
   (b) Let $R$ be a commutative ring with identity. If the polynomial $R[x]$ is a principal ideal domain, then show that $R$ is a field.

5. Let $A$ be a commutative ring with identity and let $N$ be the ideal of nilpotent elements. Show the following are equivalent:
   (a) $A$ has exactly one prime ideal.
   (b) Every element of $A$ is either a unit or nilpotent.
   (c) $A/N$ is a field.
   You may use the fact that $N$ coincides with the intersection of all prime ideals in $A$.

6. Let $M$ be a module over a commutative ring $R$. Show that $\text{Hom}_R(R, M) \simeq M$ as $R$-modules.

7. Explicitly construct a Galois extension $K$ over $\mathbb{Q}$ whose Galois group is cyclic of order 8.

8. If $f \in \mathbb{Q}[x]$ is an irreducible polynomial of odd degree with abelian Galois group then all its roots are real.