Analysis Comprehensive Exam
Spring 2021

Student Number: [ ]

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

NOTES:

- All functions in this exam are real-valued unless specified otherwise.
- The exterior Lebesgue measure of $E \subseteq \mathbf{R}^d$ is denoted by $|E|_e$, and if E is measurable then its Lebesgue measure is $|E|$.

1. Let $g_n : [0,1] \to \mathbb{R}$ be measurable for $n \geq 1$. Also let $g : [0,1] \to \mathbb{R}$. Assume that $\{g_n\}$ increase to g, that is for each $t \in [0,1]$,

$$g_1(t) \leq g_2(t) \leq g_3(t) \leq \cdots$$

and

$$\lim_{n\to\infty} g_n(t) = g(t).$$

Let

$$g^+(t) = \max\{0, g(t)\}, \quad t \in [0,1]$$

and assume that g^+ is Lebesgue integrable on $[0,1]$. Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 \ln\left(1 + e^{n g_n(t)}\right) dt$$

and justify your limit.

Definitions. (a) A linear mapping $A\colon H \to H$ is a topological isomorphism if it is a bijection and both A and A^{-1} are bounded. (b) Two inner products for a Hilbert space H are equivalent if the two norms induced from those inner products are equivalent norms for H.

2. Let H be a Hilbert space, and assume that $A\colon H \to H$ is bounded, linear, and positive definite. Prove the following statements.

(a) A is injective and has dense range.

(b) A is a topological isomorphism that maps H onto itself if and only if A is surjective.

(c) If A is surjective and positive definite, then

$$[x, y] = \langle Ax, y\rangle, \qquad \text{for } x, y \in H,$$

defines an inner product $[\cdot,\cdot]$ on H that is equivalent to the original inner product $\langle\cdot,\cdot\rangle$.

3. Let $f : [a,b] \to \mathbb{R} \cup \{-\infty, \infty\}$ be (Lebesgue) integrable. Let $\varepsilon > 0$. Show there exists a closed set F of $[a,b]$ with $|[a,b] \setminus F| < \varepsilon$, and a sequence of polynomials $\{p_n\}$ such that

$$\sup_{x\in F} |f(x) - p_n(x)| \to 0 \text{ as } n \to \infty.$$

Hint: You may assume that given $\delta > 0$, there exists a continuous function $C : [a,b] \to \mathbb{R}$ with

$$\int_a^b |f(x) - C(x)|\, dx < \delta.$$

4. Suppose that U is an unbounded open subset of $\mathbf{R}$.

   (a) For each $n \in \mathbf{N}$, set $A_n = \bigcup_{|k|>n, k\in\mathbf{Z}} U/k$, where $U/k = \{x/k : x \in U\}$. Prove that A_n is dense in $\mathbf{R}$.

   (b) Prove that the set

$$A = \big\{x \in \mathbf{R} : kx \in U \text{ for infinitely many } k \in \mathbf{Z}\big\}$$

   is dense in $\mathbf{R}$.

5. Let (S, Σ, ν) be a measure space. Assume that

$$S = \bigcup_{n=1}^{\infty} E_n$$

   where the $\{E_n\}_{n=1}^{\infty}$ are disjoint measurable sets, each with $\nu(E_n) < \infty$. Define μ on S by

$$\mu(B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\nu(B \cap E_n)}{\nu(E_n)}, \quad B \in \Sigma.$$

   (a) Show that μ is a finite measure on S.

   (b) Show that μ is absolutely continuous on S with respect to ν and ν is absolutely continuous on S with respect to μ.

   (c) Find a function $f\colon S \to \mathbb{R}$ such that for all $A \in \Sigma$,

$$\mu(A) = \int_A f \; d\nu.$$

6. Assume that $E \subseteq \mathbf{R}^d$ is measurable, with $0 < |E| < \infty$. Set

$$f(t) = |E \cap (E + t)|, \qquad \text{for } t \in \mathbf{R}^d,$$

   where $E + t = \{x + t : x \in E\}$. Show that $f(t) \to 0$ as $\|t\| \to \infty$.

7. (a) Let $E \subseteq \mathbb{R}^n$. Let $f : E \to [0, \infty)$ be measurable. Define its distribution function

$$\omega(t) = |\{f > t\}|, \quad t \geq 0.$$

   Assume that

$$\omega(t) \leq \frac{2}{1 + t^2}, \quad t \in [0, \infty).$$

   For which values of $p > 0$ is $\int_E f^p$ finite?

   (b) Find a set E and a function $f : E \to \mathbb{R}$ showing the sharpness of your range of p in part (a): that is, $\int_E f^p$ finite is precisely for the range of p you found in part (a).

8. Assume that $g\colon [a, b] \to [c, d]$ and $f\colon [c, d] \to \mathbf{R}$ are each absolutely continuous. Prove that if g is monotone increasing on $[a, b]$, then $f \circ g$ is absolutely continuous on $[a, b]$.