Student Number: [ ]

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1  2  3  4  5  6  7  8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.
1. Construct a topological space $X$ with fundamental group $\mathbb{Z}/3\mathbb{Z}$. Show that any map from $X$ to $S^1$ must be null-homotopic.

2. Let $M$ be a manifold. Define a 1-form $\lambda$ on $T^*M$ as follows. Let $\pi : T^*M \to M$ be the projection map. If $\eta \in T^*M$ and $v \in T_{\pi(\eta)}(T^*M)$ then set $\lambda_\eta(v) = \eta(d\pi_\eta(v))$. If $\alpha$ is a 1-form on $M$, then explain how $\alpha$ is a map from $M$ to $T^*M$ and show that $\alpha^*\lambda = \alpha$.

3. For which values of $a > 0$ does the hyperboloid $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a^2$ transversely in $\mathbb{R}^3$? Justify your answer.

4. Let $M$ be a compact manifold with smooth vector field $X$ and resulting 1-parameter subgroup $H^X_t \subseteq \text{Diff}(M)$; $H^X_t$ is also called a the flow of $X$. Let $f \in \text{Diff}(M)$ and let $f_*(X)$ be the pushforward of $X$. Show that

$$H^{f_*(X)}_t = f \circ H^X_t \circ f^{-1}.$$

5. Show that the set of real $2 \times 2$ matrices of rank $1$ is a 3-dimensional submanifold of the space $M_2(\mathbb{R})$ of all $2 \times 2$ matrices.

6. Let $X$ be the quotient space obtained by identifying two distinct points in $S^2$. Compute the fundamental group of $X$.

7. Say that a covering space is abelian if it is a connected, regular cover. Show that every connected CW-complex $X$ has a universal abelian cover—that is, an abelian cover that covers all other abelian covers—and that this universal cover is unique up to isomorphism of covering spaces.

8. Recall that a map $f : X \to \mathbb{R}$ defined on $X \subset \mathbb{R}^n$ is smooth if for each $x$ in $X$ there is a neighborhood $U_x$ and a smooth function $f_x : U_x \to \mathbb{R}$ such that $f = f_x$ on $X \cap U_x$. Show that if $f : X \to \mathbb{R}$ is smooth then there is a smooth extension $F : \mathbb{R}^n \to \mathbb{R}$. 

