Topology Comprehensive Exam
Spring 2020

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Construct a topological space X with fundamental group $\mathbf{Z}/3\mathbf{Z}$. Show that any map from X to S^1 must be null-homotopic.

2. Let M be a manifold. Define a 1-form λ on T^*M as follows. Let $\pi : T^*M \to M$ be the projection map. If $\eta \in T^*M$ and $v \in T_\eta(T^*M)$ then set $\lambda_\eta(v) = \eta(d\pi_\eta(v))$. If α is a 1-form on M, then explain how α is a map from M to T^*M and show that $\alpha^*\lambda = \alpha$.

3. For which values of $a > 0$ does the hyperboloid $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a^2$ transversely in $\mathbf{R}^3$? Justify your answer.

4. Let M be a compact manifold with smooth vector field X and resulting 1-parameter subgroup $H_t^X \leqslant \mathrm{Diff}(M)$; H_t^X is also called a the flow of X. Let $f \in \mathrm{Diff}(M)$ and let $f_*(X)$ be the pushforward of X. Show that
$$H_t^{f^*(X)} = f \circ H_t^X \circ f^{-1}.$$

5. Show that the set of real 2×2 matrices of rank 1 is a 3-dimensional submanifold of the space $M_2(\mathbf{R})$ of all 2×2 matrices.

6. Let X be the quotient space obtained by identifying two distinct points in S^2. Compute the fundamental group of X.

7. Say that a covering space is abelian if it is a connected, regular cover. Show that every connected CW-complex X has a universal abelian cover—that is, an abelian cover that covers all other abelian covers—and that this universal cover is unique up to isomorphism of covering spaces.

8. Recall that a map $f : X \to R$ defined on $X \subset \mathbf{R}^n$ is smooth if for each x in X there is a neighborhood U_x and a smooth function $f_x : U_x \to \mathbf{R}$ such that $f = f_x$ on $X \cap U_x$. Show that if $f : X \to \mathbf{R}$ is smooth then there is a smooth extension $F : \mathbf{R}^n \to \mathbf{R}$.