Analysis Comprehensive Exam
Fall 2019

Student Number: 

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1  2  3  4  5  6  7  8

Write only on the front side of the solution pages. A student will pass the exam if 3 problems are worked “almost perfectly” and some progress is made on a fourth problem.
1. Let $E \subset \mathbb{R}^m$ be a set of finite measure, and $f : E \rightarrow (0, \infty)$ be a measurable function. Suppose that for some $A, B > 0$ we have,

$$A \leq f(x) \leq B \quad \text{for a.e. } x \in E.$$ 

Prove that

$$\lim_{p \to 0^+} \int_E \frac{f^p - 1}{p} \, dx = \int_E (\log f) \, dx.$$ 

(b) Conclude that

$$\lim_{p \to 0^+} \left( \frac{1}{|E|} \int_E f^p \, dx \right)^{1/p} = \exp \left( \frac{1}{|E|} \int_E (\log f) \, dx \right).$$

(Hint: You may want to use a Taylor approximation of $\log (1 + pt)$ over a bounded interval)

2. Let $\{r_j\}_{j=1}^\infty$ be an enumeration of the rational numbers in $(-\infty, \infty)$. Define a set function $\mu$ on all subsets $S$ of $(-\infty, \infty)$ by

$$\mu(S) = \sum_{j: r_j \in S} \frac{1}{j^2}.$$ 

Let $\{p_j\}_{j=1}^\infty$ be an enumeration of the prime numbers. Define a set function $\nu$ on all subsets $S$ of $(-\infty, \infty)$ by

$$\nu(S) = \sum_{j: p_j \in S} \frac{(-1)^j}{2^j}.$$ 

(a) Show that $\mu, \nu$ are countably additive (signed) measures on the $\sigma$–algebra $\Sigma$ of all subsets of the real line.

(b) Show that $\nu$ is absolutely continuous with respect to $\mu$ but $\mu$ is not absolutely continuous with respect to $\nu$.

(c) Find the Lebesgue (or if you prefer Radon-Nikodym) decomposition of $\mu$ with respect to $\nu$, so that

$$\mu = \omega + \sigma$$

where $\sigma$ is singularly continuous with respect to $\nu$, while for some function $f$,

$$\omega(A) = \int_A f \, d\nu.$$ 

Also find $f$.

(d) Find the Hahn-Jordan Decomposition of $\nu$, that is

$$\nu = \nu_+ - \nu_-$$

where $\nu_-$ and $\nu_+$ are positive measures.
3. Let $0 < \beta < 1$.
   (a) Construct a measurable set $E$ in $[-1,1]$ such that
   \[
   \limsup_{\delta \to 0^+} \frac{|E \cap [-\delta,\delta]|}{2\delta} = \beta
   
   \]
   but
   \[
   \liminf_{\delta \to 0^+} \frac{|E \cap [-\delta,\delta]|}{2\delta} = 0.
   \]
   (b) What does Lebesgue's differentiation theorem say about
   \[
   \lim_{\delta \to 0^+} \frac{|E \cap [x-\delta,x+\delta]|}{2\delta}
   \]
   for a.e. $x \in E$? Just state your answer for (b), do not prove it.

4. Let $f : [0,1] \to [0,\infty)$ be a function which satisfies
   \[
   \int_0^1 e^{sf(x)} dx \leq e^{s^2}, \quad \forall s > 0.
   \]
   i. Prove that for every $t > 0$ we have
   \[
   |\{x \in [0,1] : f(x) > t\}| < e^{-t^2/4}.
   \]
   ii. Prove that $f$ is integrable, that is, prove that
   \[
   \int_0^1 f(x) dx < \infty.
   \]

5. Prove that there exists a function $f : [0,1] \to [0,\infty)$ which is absolutely continuous, strictly increasing (that is, $f(t) > f(s)$ whenever $t > s$), and such that
   \[
   |\{t \in [0,1] : f'(t) = 0\}| > 0.
   \]

6. (a) Show that the set of all irrational numbers is a $G_\delta$ set.
   (b) Show that the set $\mathbb{Q}$ of all rational numbers is an $F_\sigma$ set, but is not a $G_\delta$ set.

7. For $f \in L^1[0,1]$ denote
   \[
   \hat{f}(n) = \int_0^1 f(t)e^{-2\pi int} dt.
   \]
   Prove that for every $f \in L^1[0,1]$ we have $\hat{f}(n) \to 0$ as $|n| \to \infty$.
   (Remark: You are asked here to prove the result. Stating the fact that this is a known result will not be considered as a proof).
8. Let $H$ be a Hilbert space, and let $\{e_n\}_{n \in \mathbb{N}} \subset H$ be a sequence of elements in $H$. Assume that for every $f \in H$ we have

$$\sum_n |\langle f, e_n \rangle|^2 < \infty.$$ 

i. Prove that there exists $B > 0$ such that

$$\sum_n |\langle f, e_n \rangle|^2 \leq B \|f\|^2_H.$$ 

ii. Determine whether the following statement is true or false, and prove or disprove it accordingly: Under the above conditions we have

$$\sup_{f \in H} \sum_{n=N}^{\infty} |\langle f, e_n \rangle|^2 \to 0 \quad \text{as} \quad N \to \infty.$$