Probability Comprehensive Exam
Fall 2019

Student Number: 

Instructions: Complete 5 of the 10 problems, and circle their numbers below – the uncircled problems will not be graded.

1  2  3  4  5  6  7  8  9  10

Write only on the front side of the solution pages. A student will pass the exam if 3 problems are worked “almost perfectly” and some progress is made on a fourth problem.
1. First question.
   Show that a random variable $X$ such that
   \[ \mathbb{E}[e^{\lambda X}] \leq e^{2|\lambda|^3} \text{ for any } \lambda \in [-1, 1] \]
   satisfies $X = 0$ almost surely.

2. Assume $(X_n)_{n \geq 1}$ are iid (independent and identically distributed) random variables on some space $(\Omega, \mathcal{F}, \mathbb{P})$ with common Gumbel cumulative function given by
   \[ F(x) = e^{-e^{-x}}, x \in \mathbb{R}. \]
   Show that
   \[ \lim_{n \to \infty} \sup_n (-X_n - \ln(\ln(n))) = 0. \]

3. Let $(a_i)_{i \geq 1}$ be a sequence of positive integers such that $a_i \in \lbrack 1.01^{i-1}, 1.01^i \rbrack$ for all $i \geq 1$. Further, let $(S_n)_{n \geq 0}$ be a random walk on $\mathbb{Z}$, with $S_0 = 0$ and with $S_n = \sum_{i=1}^n X_i$, where $(X_i)_{i \geq 1}$ are mutually independent random variables, with $\mathbb{E}[X_i] = 0$ for all $i \geq 1$, and $|X_i| = a_i$, $i \geq 1$, everywhere on the probability space. Prove that the random walk $(S_n)$ is not recurrent.

4. 1. If $X$ is a random variable such that for two constants $a, b \in \mathbb{R}$, we have $a \leq X \leq b$, show that $\text{var}(X) \leq (b - a)^2/4$ and give an example of such a random variable where equality is attained.
   2. Assume that $X$ is a random variable such that $\mathbb{P}(X \leq a) = 1/2$ and $\mathbb{P}(X \geq b) = 1/2$ for some real numbers $a, b$, $a < b$. Show that $\text{var}(X) \geq (b - a)^2/4$ and give an example of such a random variable where equality is attained.

5. Let $(S_n)_{n \geq 1}$ be a simple random walk on $\mathbb{Z}$ (with $S_0 = 0$). Compute the probability mass function of the maximum of the random walk on the interval $[2n, 4n]$, i.e. compute the pmf of the variable $\xi := \max_{2n \leq i \leq 4n} S_i$. Represent the pmf as a (polynomial) function of binomial coefficients.

6. Assume $(X_n)_{n \geq 1}$ is a sequence of iid positive random variables. Show that
   \[ \frac{X_1 + X_2^2 + \cdots + X_n^n}{n} \xrightarrow{n \to \infty} 1 \text{ if and only if } X_1 = 1 \text{ almost surely.} \]
7. (Modified Polya’s urn) Consider the following discrete time process. Before time one, we have one white and one black ball in the urn. At time \( k \) (\( k \geq 1 \)), we pick a ball from the urn uniformly at random with replacement, and add to the urn a ball of the color opposite to the color of the ball we have picked. Thus, at each step the number of balls in the urn increases by one. Let \( X_n \) be the proportion of white balls in the urn right after the \( n \)-th step. Show that \( (X_n) \) converges to \( 1/2 \) almost surely.

8. Let \( b_1, b_2, \ldots \) be a sequence of mutually independent Bernoulli(1/2) random variables, and let \( m \) be a fixed positive integer. Define a process \( (X_i)_{i \geq 0} \) as follows. Set \( X_0 = 1 \), and define \( X_i \) recursively as

\[
X_i = \begin{cases} 
X_{i-1} + \frac{1}{m}(2b_i - 1), & \text{if } X_{i-1} \geq 1/m; \\
0, & \text{otherwise}.
\end{cases}
\]

Show that \( (X_n) \) converges to zero almost everywhere.

9. Let \( m \geq 2 \) be a positive integer, and let \( b_1, b_2, \ldots \) be mutually independent Bernoulli(1/2) variables. Consider the following Markov chain \( (X^n) \) in \( \mathbb{R}^m \). Let \( X^0 \) be a fixed 0/1-vector in \( \mathbb{R}^m \). Next, given \( X^{i-1} = (x_1^{i-1}, x_2^{i-1}, \ldots, x_m^{i-1}) \), we set \( X^i = (x_1^i, x_2^i, \ldots, x_m^i) \) to be the 0/1-vector such that

\[
\sum_{j=1}^{m} 2^{n-j} x_j^i - \sum_{j=1}^{m} 2^{n-j} x_j^{i-1} = \begin{cases} 
b_i(1 - 2^m), & \text{if } x_1^{i-1} = x_2^{i-1} = \cdots = x_m^{i-1} = 1; \\
b_i, & \text{otherwise}.
\end{cases}
\]

(Above, “\( i-1 \)” and “\( i \)” are upper indices, not powers)

1) Prove that the Markov chain \( (X^n) \) converges in distribution to the uniform distribution on the set \( \{0, 1\}^m \).

2) Recall that the mixing time \( t_{mix} \) is defined as the smallest integer such that for all \( n \geq t_{mix} \) the total variation distance between the distribution of \( X^n \) and the stationary distribution is at most 1/4. Show that the mixing time \( t_{mix} \) of \( (X^n) \) satisfies \( c2^{2m} \leq t_{mix} \leq C2^{2m} \) for some universal constants \( c, C > 0 \). You should not use as a blackbox “known” estimates on mixing times, please outline the proof.

10. On a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) assume we have three random variables \( X, Y, Z \) independent and uniform on \([0, 1]\). Compute \( \mathbb{E}[\min\{X, Y, Z\}] \).